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WENTWORTH & HILL'S

EXERCISE MANUALS.

No. III.

GEOMETRY.

BOSTON, U.S.A.:
PUBLISHED BY GINN & COMPANY.
1889.

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Prof. Ephraim Emerton.

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PREFACE.

THE aim of wisely-directed mathematical teaching is to cultivate the reasoning faculty, not the memory; and the true test of mathematical training is the *power* which the learner has acquired over original problems.

This truth is very generally recognized in teaching Arithmetic and Algebra, but very generally ignored in teaching Geometry. There are, however, many signs that a change in the method of teaching Geometry is taking place in this country. Most of the recent text-books of Geometry contain exercises designed to stimulate original thought; most papers now set in this subject for admission to college require more or less original work; and intelligent teachers are demanding a collection of suitable geometrical exercises.

The present work has been prepared to meet this demand, with the hope that it will promote a much-needed reform. It is the first work in the English language, so far as the authors know, in which the subject of geometrical exercises is systematically treated. The materials for the work have been drawn chiefly from French and German sources. The arrangement and mode of treatment are such as seem best adapted to meet the wants of American schools.

It is not intended that each student should try to work every exercise in the book, but the teacher should assign particular problems to separate students, selecting them with special reference to the capacity and skill of each pupil. With care on the part of the teacher, pupils will gain the mastery over problems in Geometry as readily as they do over problems in Algebra, and precisely in the

same way; namely, by working them out. It is necessary at first to give easy problems; but the doing of easy problems prepares the way for harder ones and still harder.

The exercises here given consist of a great number of easy problems for beginners, and enough harder ones for more advanced scholars. The exercises in each section are carefully graded, and some of the more difficult sections can be omitted without destroying the unity of the work. The book can be used in connection with any text-book on Geometry, as soon as the geometrical processes of reasoning are well understood.

A Syllabus of Geometry is given, not only for reference, but with a view of making the book by itself convenient for reviewing the study of Geometry. Lessons can be assigned consisting partly of book-work taken from the Syllabus, and partly of original work, and the two parts can be so fitted to each other that a thorough knowledge of the book-work will be necessary in order to do the original work, and the doing of the original work will firmly fix in the mind the principles involved in the book-work.

Any corrections or any suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.
G. A. HILL.

PHILLIPS EXETER ACADEMY,
September, 1884.

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SYLLABUS

OF

PLANE GEOMETRY.

AXIOMS.

1. Magnitudes which can be made to coincide are equal.
2. Two magnitudes, each equal to a third, are equal to each other.
3. If equals are added to equals, the sums are equal.
4. If equals are taken from equals, the remainders are equal.
5. If equals are added to unequals, the sums are unequal in the same sense.
6. If equals are taken from unequals, the remainders are unequal in the same sense.
7. If unequals are taken from equals, the remainders are unequal in the opposite sense.
8. The whole is equal to the sum of its parts.
9. Through two points only one straight line can be drawn.
10. A straight line is the shortest line between two points.
11. Through a point not in a straight line only one parallel to the line can be drawn.

BOOK I.

THE STRAIGHT LINE.

Definitions.

12. Body, surface, line, point, straight line, curved line, broken line, plane surface or plane, curved surface, figure, plane figure, similar figures, equivalent figures, equal figures, test of the equality of geometrical figures, method of superposition.

Object of Geometry, Plane Geometry, Solid Geometry, axiom, theorem, corollary, scholium, problem, postulate, proposition, hypothesis, conclusion, proof, converse theorem, contrary theorem.

Comparison of lines as regards magnitude, linear units, length of a line, addition, subtraction, multiplication, and division of lines.

Angle, its sides, its vertex, naming of an angle, straight angle, right angle, acute angle, obtuse angle, generation of an angle, comparison of angular magnitudes, division of the right angle into degrees, minutes, and seconds, adjacent angles, vertical angles, supplementary angles, complementary angles, bisector of an angle, perpendicular lines, oblique lines.

Parallel lines, secant, alternate-interior angles, alternate-exterior angles, exterior-interior angles, interior and exterior angles on the same side of the secant.

Polygon, its sides, its angles, its vertices, its parts, triangle, quadrilateral, pentagon, hexagon, octagon, decagon, dodecagon, perimeter, diagonal, exterior angles, convex

polygon, reëntrant angles, equilateral polygon, equiangular polygon, mutually equiangular polygons, mutually equilateral polygons, homologous vertices, angles, and sides, equal polygons.

Equilateral triangle, isosceles triangle, scalene triangle, right triangle, acute triangle, obtuse triangle, hypotenuse, legs, base, vertex, altitude.

A triangle has three altitudes because each side may be taken as base. The three lines drawn from the vertices to the opposite sides and bisecting the angles are called the bisectors. The three lines drawn from the vertices to the middle points of the opposite sides are called the medians. In the isosceles triangle the two equal sides are called the legs; the other side is called the base; and the vertex of the angle opposite to the base is called the vertex of the triangle.

Parallelogram, square, rectangle, rhombus, rhomboid, trapezoid.

The two parallel sides of a trapezoid are called the bases; the other sides, the legs; the line joining the middle points of the legs, the median. An isosceles trapezoid is a trapezoid having equal legs.

A deltoid is a quadrilateral having two adjacent sides equal, and the other two sides also equal.

The centre of a parallelogram is the intersection of its diagonals.

Angles, Perpendiculars, and Parallels.

13. *Theorem.* All straight angles are equal.

14. *Corollary* I. All right angles are equal.

15. *Corollary* II. The sum of the adjacent angles formed by producing one side of an angle from the vertex is equal to 180°.

16. *Corollary* III. The sum of all the consecutive angles which can be formed on one side of a straight line at the same point in the line is equal to 180°.

17. *Corollary* IV. The sum of all the consecutive angles which can be formed about a point is equal to 360°.

18. *Theorem.* If the sum of two adjacent angles is equal to 180°, their exterior sides form a straight line.

19. *Theorem.* Vertical angles are equal.

20. *Theorem.* Only one perpendicular can be erected at a given point in a straight line.

21. *Theorem.* Only one perpendicular can be drawn from a given point to a straight line.

22. *Theorem.* Two straight lines, each perpendicular to the same straight line, are parallel.

23. *Theorem.* If a straight line is perpendicular to one of two parallels, it is also perpendicular to the other.

24. *Theorem.* If two straight lines are each parallel to a third line, they are parallel to each other.

25. *Theorem.* If two parallels are cut by a secant:

(i.) the alternate-interior angles are equal;

(ii.) the alternate-exterior angles are equal;

(iii.) the exterior-interior angles are equal;

(iv.) the interior or the exterior angles on the same side of the secant are supplementary.

26. *Theorem.* If *any one* of the four conditions in No. 25 is satisfied, the two lines cut by the secant are parallel.

27. *Theorem.* Two angles whose sides are respectively parallel are equal or supplementary: equal, if both are acute or both obtuse; supplementary, if one is acute and the other obtuse.

28. *Theorem.* Two angles whose sides are respectively perpendicular are equal or supplementary: equal, if both are acute or both obtuse; supplementary, if one is acute and the other obtuse.

29. *Theorem.* The sum of the three angles of a triangle is equal to 180°.

30. *Corollary* I. If two triangles have two angles equal, each to each, the third angles are also equal.

31. *Corollary* II. The sum of the acute angles of a right triangle is equal to 90°.

32. *Theorem.* An exterior angle of a triangle is equal to the sum of the two opposite interior angles.

33. *Theorem.* The sum of the angles of a quadrilateral is equal to 360°.

34. *Theorem.* The sum of the angles of a polygon of n sides is equal to $(n-2) \times 180°$.

35. *Theorem.* The sum of the exterior angles of a convex polygon of n sides, formed by producing each side in succession, is equal to 360°.

Triangles.

36. *Theorem.* Two triangles are equal if they have a side and the two adjacent angles equal, each to each.

37. *Corollary.* Two right triangles are equal if they have a side and an acute angle of the one equal to a corresponding side and acute angle of the other.

38. *Theorem.* Two triangles are equal if they have two sides and the included angle equal, each to each.

39. *Corollary.* Two right triangles are equal if their legs are equal, each to each.

40. *Theorem.* In an isosceles triangle the angles opposite the equal sides are equal.

41. *Corollary.* An equilateral triangle is equiangular.

42. *Theorem.* If in a triangle two angles are equal, the opposite sides are equal, and the triangle is isosceles.

43. *Corollary.* An equiangular triangle is also equilateral.

44. *Theorem.* In an isosceles triangle the median drawn from the vertex to the base is perpendicular to the base and bisects the angle at the vertex.

45. *Theorem.* Two triangles are equal if their three sides are equal, each to each.

46. *Theorem.* Two right triangles are equal if they have a leg and the hypotenuse equal, each to each.

47. *Theorem.* Of two angles of a triangle that is the greater which is opposite the greater side.

48. *Theorem.* Of two sides of a triangle that is the greater which is opposite the greater angle.

49. *Theorem.* If two triangles have two sides equal, each to each, but the included angles unequal, the third side is greater in that triangle which has the greater angle.

50. *Theorem.* If two triangles have two sides equal, each to each, but the third sides unequal, the angle included by the equal sides is greater in that triangle which has the greater side.

PARALLELOGRAMS.

51. *Theorem.* The diagonal of a parallelogram divides it into two equal triangles.

52. *Corollary* I. The opposite sides and the opposite angles of a parallelogram are equal.

53. *Corollary* II. Parallels between parallels are equal.

54. *Corollary* III. Parallels are everywhere equally distant.

55. *Theorem.* The diagonals of a parallelogram bisect each other.

56. *Theorem.* A quadrilateral is a parallelogram if:

(i.) the opposite sides are equal;

(ii.) the opposite angles are equal;

(iii.) two opposite sides are equal and parallel;

(iv.) the diagonals mutually bisect each other.

57. *Theorem.* The diagonals of a rectangle are equal.

58. *Theorem.* The diagonals of a rhombus are perpendicular to each other.

59. *Corollary.* The diagonals of a square are equal and perpendicular to each other.

60. *Theorem.* Two parallelograms are equal if they have two adjacent sides and the included angle equal, each to each.

Theorems Relating to Lengths and Distances.

61. *Theorem.* Each side of a triangle is less than the sum of the other two sides, and greater than their difference.

62. *Theorem.* The sum of two straight lines, drawn from a point to the ends of a straight line, is greater than the sum of any other two straight lines similarly drawn but included by them.

63. *Theorem.* A perpendicular is the shortest distance from a point to a straight line.

64. *Theorem.* Two oblique lines, drawn from a point in a perpendicular so as to cut off equal distances from the foot of the perpendicular, are equal; and of two oblique lines which cut off unequal distances, the more remote is the greater.

65. *Corollary.* Only two equal straight lines can be drawn from a point to a straight line.

66. *Theorem.* Two equal oblique lines cut off equal distances from the foot of the perpendicular; and of two unequal oblique lines, the greater cuts off the greater distance from the foot of the perpendicular.

67. *Theorem.* Every point in the perpendicular, erected at the middle point of a straight line, is equidistant from the ends of the line, and every point not in the perpendicular is unequally distant from the ends of the line.

68. *Corollary.* Two points equidistant from the ends of a straight line determine the perpendicular through the middle point of the line.

69. *Theorem.* Every point in the bisector of an angle is equidistant from the sides of the angle, and every point not in the bisector is unequally distant from the sides.

70. *Theorem.* The line joining the middle points of two sides of a triangle is parallel to the third side and equal to half of the third side.

71. *Theorem.* A straight line drawn parallel to the base of a triangle, bisecting one of the sides, bisects the other also; and the part intercepted between the two sides is equal to half the base.

72. *Theorem.* The median of a trapezoid is parallel to the bases and equal to half their sum.

73. *Theorem.* Equidistant parallels divide every secant into equal parts.

BOOK II.

THE CIRCLE.

The Circle and Straight Lines.

74. *Definitions.* Circumference, circle, radius, diameter, arc, chord, semi-circumference, segment, sector, quadrant, semicircle, secant, tangent, point of contact, chord of contact, inscribed polygon, circumscribed circle, circumscribed polygon, inscribed circle.

A chord quadrilateral is a quadrilateral about which a circle can be circumscribed.

A tangent quadrilateral is a quadrilateral in which a circle can be inscribed.

75. *Theorem.* A straight line cannot meet a circumference in more than two points.

76. *Theorem.* Every diameter divides a circle and its circumference into two equal parts.

77. *Theorem.* The diameter is greater than any other chord.

78. *Theorem.* Circles having equal radii or equal diameters are equal; and conversely.

79. *Theorem.* In the same circle, or in equal circles, equal arcs are subtended by equal chords; and conversely.

80. *Theorem.* The radius perpendicular to a chord bisects the chord and the arcs subtended by the chord.

81. *Theorem.* The perpendicular erected at the middle point of a chord passes through the centre and bisects the arc subtended by the chord.

82. *Scholium.* The line mentioned in the last two propositions satisfies four conditions: (i.) it passes through the centre; (ii.) it is perpendicular to the chord; (iii.) it bisects the chord; (iv.) it bisects the arc subtended by the chord. Any two of these conditions determine this line. Since two things can be selected from four things in six different ways, there are altogether six different theorems respecting this line.

83. *Theorem.* Through three points not in a straight line one circumference can be drawn, and only one.

84. *Corollary.* If two circumferences have three points common, they coincide throughout.

85. *Theorem.* In the same circle, or in equal circles, equal chords are equally distant from the centre; and conversely.

86. *Theorem.* In the same circle, or in equal circles, of two unequal chords the greater is nearer the centre; and conversely.

87. *Theorem.* A tangent to a circle is perpendicular to the radius drawn to the point of contact.

88. *Theorem.* A perpendicular to a tangent erected at the point of contact passes through the centre.

89. *Theorem.* The tangents to a circle drawn through an exterior point are equal, and make equal angles with the line joining the point to the centre.

Variables and their Limits.

90. *Definitions.* Measure of a quantity, common measure of two quantities, ratio of two quantities, antecedent, consequent, commensurable and incommensurable ratios, approximate value, constant, variable, limit of a variable.

91. *Proposition* I. If two variables are constantly equal, and tend each towards a limit, these limits are also equal.

92. *Proposition* II. The limit of the algebraic sum of two or more variables is equal to the algebraic sum of their limits.

93. *Proposition* III. The limit of the product of two or more variables is equal to the product of their limits.

94. *Proposition* IV. The limit of the ratio of two variables is equal to the ratio of their limits.

The Circle and Angles.

95. *Definitions.* Angle at the centre, inscribed angle, angle standing on a chord, angle inscribed in a segment.

96. *Theorem.* In the same circle, or in equal circles, equal angles at the centre intercept equal arcs; and conversely.

97. *Theorem.* In the same circle, or in equal circles, the ratio of two angles at the centre is equal to the ratio of the intercepted arcs.

98. *Corollary* I. An angle has the same numerical measure as an arc described from its vertex as centre and intercepted by its sides, if the unit of arc is defined to be the arc intercepted by the sides of unit angle.

99. *Theorem.* An inscribed angle is measured by half the intercepted arc.

100. *Corollary* I. All angles inscribed in the same segment are equal.

101. *Corollary* II. Every angle inscribed in a semicircle is a right angle.

102. *Corollary* III. Every angle inscribed in a segment greater than a semicircle is acute, and every angle inscribed in a segment less than a semicircle is obtuse.

103. *Theorem.* The angle formed by a tangent and a chord is measured by half the intercepted arc.

104. *Theorem.* An angle which has its vertex within the circle is measured by half the sum of the intercepted arcs.

105. *Theorem.* An angle which has its vertex without the circle is measured by half the difference of the intercepted arcs.

106. *Theorem.* The opposite angles of an inscribed quadrilateral are supplementary.

107. *Theorem.* If two opposite angles of a quadrilateral are supplementary, the quadrilateral is a chord quadrilateral.

Two Circles.

108. *Definitions.* Concentric circles, line of centres, external contact, internal contact, exterior tangents, interior tangents.

109. *Theorem.* If two circles cut each other, the line of centres is perpendicular to the common chord and bisects it.

110. *Theorem.* If two circles touch each other, the line of centres passes through the point of contact.

Problems.

111. *Instruments.* Ruler, compasses, set-square, protractor.

A problem does not belong to Elementary Geometry unless it can be solved with ruler and compasses only.

112. To erect a perpendicular at a given point in a given line.

113. To drop a perpendicular from a given point to a given line.

114. To bisect a given line.

115. To bisect a given arc.

116. To bisect a given angle.

117. At a given point in a given line to construct an angle equal to a given angle.

118. Through a given point to draw a parallel to a given line.

119. From a point without a given line to draw a line making a given angle with this line.

120. Two angles of a triangle being given, to find the third angle.

121. To construct a triangle, given two sides and the included angle.

122. To construct a triangle, given one side and the two adjacent angles. When is the problem impossible?

123. To construct a triangle, given one side, one adjacent angle, and the opposite angle.

124. To construct a triangle, given the three sides. When is the problem impossible?

125. To construct a triangle, given two sides and the angle opposite one of these sides. When will the problem have two solutions? one solution? no solution?

126. To divide a given line into any number of equal parts.

127. To find the ratio of two given commensurable straight lines.

128. To find the centre of a given circumference.

129. To describe a circumference through three given points; or, to circumscribe a circle about a given triangle.

130. To inscribe a circle in a given triangle.

131. Through a given point to draw a tangent to a given circle. When will the problem have one solution? two solutions? no solution?

132. To draw a common tangent to two given circles. When will the problem have four solutions? three solutions? two solutions? one solution? no solution?

133. To construct upon a given line a segment capable of containing a given angle.

BOOK III.

SIMILAR FIGURES.

Theory of Proportions.

134. *Definitions.* Proportion, terms of a proportion, extremes and means, inverse or reciprocal proportion.

135. *Theorem.* If $a : b = c : d$, then $ad = bc$.

136. *Theorem.* If $ad = bc$, then $a : b = c : d$.

137. *Theorems.* If $a : b = c : d$, then:

(i.) $a : c = b : d$;

(ii.) $a \pm c : b \pm d = a : b = c : d$.

(iii.) $a \pm b : c \pm d = a : c = b : d$;

(iv.) $a + c : a - c = b + d : b - d$;

(v.) $a + b : a - b = c + d : c - d$;

(vi.) $ma : mb = nc : nd$;

(vii.) $a^n : b^n = c^n : d^n$.

Proportional Lines.

138. *Definitions.* Proportional lines, third proportional, mean proportional, extreme and mean ratio, segments of a line.

139. *Theorem.* A line parallel to one side of a triangle divides the other two sides into proportional parts; and these two sides have the same ratio as two corresponding segments.

140. *Theorem.* A line which divides two sides of a triangle into proportional parts is parallel to the third side.

141. *Theorem.* Parallels cut proportional parts from any two straight lines.

142. *Theorem.* Lines meeting in a common point divide parallels into proportional parts.

143. *Theorem.* The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

144. *Theorem.* The bisector of an exterior angle of a triangle divides the opposite side produced into segments proportional to the other two sides.

Similar Triangles.

145. *Definitions.* Similar triangles, homologous angles, vertices, sides, bases, and altitudes.

146. *Theorem.* Two triangles are similar if they are equiangular with respect to each other.

147. *Corollary* I. Two triangles are similar if they have two angles equal, each to each.

148. *Corollary* II. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

149. *Theorem.* Two triangles are similar if they have an angle of one equal to an angle of the other and the including sides proportional.

150. *Corollary.* Two right triangles are similar if their legs are proportional.

151. *Theorem.* Two triangles are similar if their sides, taken in order, are proportional.

152. *Theorem.* The homologous altitudes of two similar triangles are proportional to any two homologous sides.

Similar Polygons.

153. *Definitions.* Similar polygons, homologous angles, vertices, sides, and diagonals.

154. *Theorem.* Two similar polygons are divisible into the same number of triangles, similar each to each, and similarly placed.

155. *Theorem.* Two polygons are similar if they are divisible into the same number of triangles, similar each to each, and similarly placed.

156. *Theorem.* The perimeters of two similar polygons are proportional to any two homologous sides.

Numerical Properties of Figures.

157. *Definition.* The projection of a line.

158. *Theorem.* In a right triangle:

(i.) the altitude upon the hypotenuse divides the triangle into two right triangles, similar to the given triangle and to each other;

(ii.) the altitude is a mean proportional between the segments of the hypotenuse;

(iii.) each leg is a mean proportional between the hypotenuse and the adjacent segment;

(iv.) the squares of the legs are proportional to the adjacent segments;

(v.) the product of the two legs is equal to the product of the hypotenuse and the altitude.

159. *Theorem.* A perpendicular dropped from a point in the circumference of a circle to a diameter is a mean proportional between the segments of the diameter.

160. *Theorem.* A chord drawn from the end of a diameter is a mean proportional between the diameter and the projection of the chord upon the diameter.

161. *Theorem.* In a right triangle the square of the hypotenuse is equal to the sum of the squares of the two legs.

162. *Theorem* In a triangle the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of these sides and the projection of the other upon it.

163. *Theorem.* In an obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of these sides and the projection of the other upon it.

164. *Theorem.* If through a fixed point within a circle a chord is drawn, the product of the two segments of the chord is constant in whatever direction the chord is drawn.

165. *Theorem.* If from a fixed point without a circle a secant is drawn, the product of the entire secant and the external segment is constant in whatever direction the secant is drawn.

166. *Theorem.* If from a point without a circle a tangent and a secant are drawn, the tangent is a mean proportional between the entire secant and the external segment.

Problems.

167. To divide a given line into parts proportional to any number of given lines.

168. To find the fourth proportional to three given lines.

169. To find the third proportional to two given lines.

170. To find the mean proportional between two given

171. To divide one side of a triangle into two parts proportional to the other two sides.

172. To divide a given line in extreme and mean ratio.

173. To construct upon a given line a triangle similar to a given triangle.

174. To construct upon a given line a polygon similar to a given polygon.

BOOK IV.

EQUIVALENT FIGURES.

AREAS.

175. *Definitions.* Equivalent figures, area of a figure, units of area, transformation of a figure.

176. *Theorem.* Two rectangles having equal bases are to each other as their altitudes; and two rectangles having equal altitudes are to each other as their bases.

177. *Theorem.* Any two rectangles are to each other as the products of their bases and altitudes.

178. *Theorem.* Area of a rectangle = base × altitude.

179. *Theorem.* Area of a square = square of one side.

180. *Theorem.* Area of a parallelogram = base × altitude.

181. *Theorem.* Two parallelograms having equal bases are to each other as their altitudes; and two parallelograms having equal altitudes are to each other as their bases.

182. *Theorem.* Area of a triangle = ½ base × altitude.

183. *Theorem.* Two triangles having equal bases are to each other as their altitudes; and two triangles having equal altitudes are to each other as their bases.

184. *Theorem.* Area of a trapezoid = ½ sum of the bases × altitude.

185. *Theorem.* Area of a trapezoid = median × altitude.

186. *Theorem.* Area of a polygon circumscribed about a circle = ½ perimeter × radius of the circle.

Comparison of Areas.

187. *Theorem.* The areas of two triangles having an angle of one equal to an angle of the other are to each other as the rectangles of the sides including the equal angles.

188. *Theorem.* Similar triangles are to each other as the squares upon the homologous sides.

189. *Theorem.* Similar polygons are to each other as the squares upon the homologous sides or homologous diagonals.

190. *Theorem.* In a right triangle the square upon the hypotenuse is equivalent to the sum of the squares upon the two legs.

191. *Theorem.* In a triangle the square upon a side opposite an acute angle is equivalent to the sum of the squares upon the other two sides, diminished by twice the rectangle of one of these sides and the projection of the other upon it.

192. *Theorem.* In an obtuse triangle the square upon the side opposite the obtuse angle is equivalent to the sum of the squares upon the other two sides, increased by twice the rectangle of one of these sides and the projection of the other upon it.

Problems.

193. To find the area of a given polygon.

194. To construct a square equivalent to the sum of two or more given squares.

195. To construct a square equivalent to the difference of two given squares.

196. To construct a polygon similar to two or more given similar polygons and equivalent to their sum.

197. To construct a polygon similar to two given similar polygons and equivalent to their difference.

198. To construct a square which shall have a given ratio to a given square.

199. To construct a polygon similar to a given polygon, and having a given ratio to it.

200. To construct a triangle equivalent to a given polygon.

201. To construct a square equivalent to a given parallelogram.

202. To construct a square equivalent to a given triangle.

203. To construct a square equivalent to any given polygon.

204. To construct a polygon similar to a given polygon, and equivalent to another given polygon.

BOOK V.

REGULAR FIGURES.

Regular Polygons.

205. *Definitions.* Regular polygons, inscribed and circumscribed regular polygons.

206. *Theorem.* A circle may be circumscribed about every regular polygon and also inscribed in every regular polygon.

207. *Definitions.* Centre of a regular polygon, radius, apothem, angle at the centre.

208. *Theorem.* Each angle at the centre of a regular polygon of n sides $= \frac{360°}{n}$.

209. *Theorem.* Each angle of a regular polygon is the supplement of the angle at the centre.

210. *Theorem.* If a circumference is divided into equal parts, the chords joining the consecutive points of division form an inscribed regular polygon; and the tangents drawn through these points form a circumscribed regular polygon.

211. *Theorem.* Two regular polygons of the same number of sides are similar.

212. *Theorem.* The perimeters of two regular polygons of the same number of sides have the same ratio as their radii, or as their apothems.

213. *Theorem.* Area of a regular polygon $= \frac{1}{2}$ perimeter $\times$ apothem.

Circumference and Area of the Circle.

The letter r is used to denote the radius of a circle.

214. *Definitions.* Similar arcs, sectors, and segments.

215. *Theorem.* The limit of the perimeter of a circumscribed regular polygon, or of an inscribed regular polygon, when the number of sides is indefinitely increased, is the circumference of the circle.

216. *Theorem.* Two circumferences have the same ratio as their radii, or their diameters.

217. *Theorem.* The ratio of the circumference of a circle to its diameter is constant for all circles. This ratio is denoted by the Greek letter π.

218. *Theorem.* Circumference of a circle $= 2\pi r$.

219. *Theorem.* Similar arcs are as their radii.

220. *Theorem.* Arc of $n° = \frac{n}{360} \times 2\pi r$.

221. *Theorem.* Area of a circle $= \pi r^2$.

222. *Theorem.* Two circles are to each other as the squares of their radii, or of their diameters.

223. *Theorem.* Similar sectors are to each other as the squares of their radii.

224. *Theorem.* Sector of $n^\circ = \frac{n}{360} \times \pi r^2$.

225. *Theorem.* Similar segments are to each other as the squares of their radii.

Problems.

226. To inscribe a square in a given circle.

227. To inscribe a regular hexagon and an equilateral triangle in a given circle.

228. To inscribe a regular decagon and a regular pentagon in a given circle.

229. To inscribe in a circle a regular polygon of 15 sides.

230. To inscribe in a given circle a regular polygon having double the number of sides of a given inscribed regular polygon.

231. *Scholium.* Nos. 226–230, enable us to inscribe in a circle regular polygons, the number of whose sides is represented by one of the terms of the following four series:

3, 6, 12 ..... 3×2^n; 4, 8, 16 ..... 4×2^n;

5, 10, 20 ..... 5×2^n; 15, 30, 60 ..... 15×2^n.

232. To circumscribe about a given circle a regular polygon similar to a given inscribed regular polygon.

233. To construct a regular polygon, given one side and the number of sides.

234. Given the side a of a regular polygon inscribed in a circle, to compute the side b of an inscribed regular polygon having double the number of sides.

$$b = \sqrt{2r - \left(r - \sqrt{r^2 - \frac{a^2}{4}}\right)}.$$

235. To compute approximately the value of π.

Exercise Manual in Geometry.

CHAPTER I.

THE STRAIGHT LINE.

§ 1. Questions and Numerical Exercises.

1. An angle is equal to $28° 53' 36''$; find an angle six times as large.

2. What angle do the hands of a clock make at 5 o'clock?

3. What angle is described by the minute-hand of a clock in 15 minutes? What angle is described by the hour-hand in the same time?

4. Find the values of two adjacent supplementary angles, if one is 9 times as large as the other.

5. Find the supplement of $32°$, and the complement of $88° 12' 16''$.

6. What is the supplement of the complement of $40°$? the complement of the supplement of $91° 1' 1''$?

7. Around a point as common vertex are arranged 8 angles, each angle being $5° 10'$ greater than the one preceding it in order. Find all the angles.

8. Through the vertex of a right angle a line is drawn outside the angle. What is the sum of the two acute angles thereby formed?

9. One acute angle of a right triangle is $51° 32'$; find the other acute angle.

10. Of the angles of a triangle the second is twice the first, and the third three times the second. Find all the angles.

11. If in a right triangle one acute angle is $A°$ greater than the other, find the value of each angle.

12. If A, B, C, denote the angles of a triangle, find the angles formed by (i.) the bisectors of A and B; (ii.) the altitudes upon AC and BC; (iii.) the bisector of A, and the altitude upon BC.

13. If three angles of a quadrilateral are right angles, what is the value of the fourth angle?

14. What is the sum of the angles of an octagon?

15. Find each angle of an equiangular decagon.

16. Find the sum of the acute angles of a starred pentagon; also the sum of all the interior angles.

The starred pentagon or *pentagram* is formed by producing the sides of an equiangular pentagon until they meet, and then erasing the sides of the pentagon.

17. Find the sum of the angles of a polygon of 20 sides.

18. The sum of the angles of a polygon is 48 right angles; find the number of sides.

19. In what polygon is the sum of the angles three times as great as in a pentagon?

20. Make a quadrilateral having the greatest possible number of obtuse angles.

21. Make a hexagon having the greatest possible number of reëntrant angles.

22. How many diagonals can be drawn from one vertex of a decagon? How many from all the vertices?

23. How many different diagonals can be drawn in a polygon of 20 sides?

24. If one of the base angles of an isosceles triangle $= 25°$, find the angle at the vertex.

25. What is the value of each acute angle in an isosceles right triangle?

26. Find the angle formed by the bisectors of two angles in an equilateral triangle.

27. Find the angles of an isosceles triangle if a base angle is double the vertex angle.

28. The bisector of a base angle in an isosceles triangle makes, with the opposite leg, the angle $52° 15'$; find all the angles of the triangle.

29. If the angle at the vertex of an isosceles triangle is $A°$, find the angle formed by the bisectors of the base angles.

30. Find the angles of an isosceles triangle if the altitude is equal to half the base.

31. In an isosceles triangle having a given base, between what limits must the value of a leg lie?

32. If two equal right triangles, each having the acute angles $30°$ and $60°$, are placed with their longer legs coinciding, and the other legs extending in opposite directions, what kind of a triangle will be formed?

33. In a right triangle one leg is equal to half the hypotenuse; find the acute angles. (See Ex. 32.)

34. If the acute angles of a right triangle are 30° and 60°, what angle is formed by the bisectors of the acute angles?

35. Make a triangle, and then draw the three altitudes. If two of the altitudes lie without the triangle, what must be true of one of the angles?

36. By drawing an altitude of a triangle (and, if necessary, producing the base) two right triangles are formed; what relation exists between these two triangles and the original triangle, (i.) when the altitude lies within the original triangle, (ii.) when it lies without?

37. Can a triangle be made, having for sides 3 feet, 5 feet, and 12 feet? 4 feet, 3 feet, and 7 feet? 2 feet, 7 feet, and 3 feet?

38. Two sides of a triangle are 12 inches and 15 inches; between what limits must the third side lie?

39. If one angle of a parallelogram is 75°, find the other angles.

40. The difference between two adjacent angles of a parallelogram is 90°; find all the angles.

41. Into what figures is a rhombus divided by drawing one diagonal? two diagonals?

42. What is the sum of the two angles of a trapezoid adjacent to one leg?

43. If one angle of an isosceles trapezoid is 45°, find the other angles.

44. If the bases of a trapezoid are 3 inches and 7 inches in length, what is the length of the median?

45. The legs of a trapezoid are perpendicular to each other, and one angle of the trapezoid is 60°; find the other three angles.

§ 2. Theorems.

1. The bisectors of two supplementary adjacent angles are perpendicular to each other.

2. The bisectors of two vertical angles form one straight line.

3. The bisectors of two alternate-interior angles formed by parallel lines are parallel.

4. The bisectors of the acute angles of a right triangle form an angle of 135°.

5. The bisector of the angle at the vertex of an isosceles triangle bisects the base and is perpendicular to the base.

6. The perpendicular drawn from the vertex of an isosceles triangle to the base bisects the base and also the angle at the vertex.

7. The perpendicular erected at the middle point of the base of an isosceles triangle passes through the vertex and bisects the angle at the vertex.

8. The bisectors of the base angles of an isosceles triangle form with the base another isosceles triangle.

9. If the angle at the vertex of an isosceles triangle is 36°, the bisector of a base angle divides the triangle into two isosceles triangles. Find the lengths of all the lines in the figure, if a leg of the given triangle is denoted by a and the base by b.

10. The altitudes upon the legs of an isosceles triangle are equal. What follows as to the three altitudes of an equilateral triangle?

11. The medians drawn to the legs of an isosceles triangle are equal.

12. The bisectors of the base angles of an isosceles triangle are equal.

13. If the angle at the vertex of an isosceles triangle is a right angle, what relation exists between the base and the altitude? Prove.

14. The perpendiculars dropped from the middle point of the base of an isosceles triangle to the legs are equal.

15. If a leg of an isosceles triangle is produced from the vertex by its own length, and the extremity joined to the extremity of the base, the joining line is perpendicular to the base.

16. If through any point in the base of an isosceles triangle parallels to the legs are drawn, a parallelogram will be formed, having its perimeter equal to the sum of the legs of the triangle.

17. The sum of the perpendiculars dropped from a point in the base of an isosceles triangle to the legs is constant and equal to the altitude upon a leg.

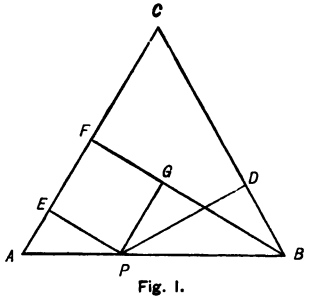


Fig. 1.

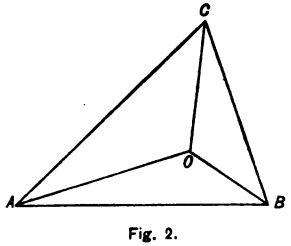


Fig. 2.

Let (*Fig.* 1) *PD* and *PE* be the two perpendiculars, *BF* the altitude upon the leg *AC*. Draw *PG* ⊥ *BF*, and prove that the triangles *PBG* and *PBD* are equal.

18. The sum of the perpendiculars dropped from any point within an equilateral triangle to the three sides is equal to the altitude.

Draw through the given point a parallel to one side of the triangle; this reduces the theorem to Ex. 17.

19. The sum of the lines which join a point within a triangle to the three vertices is less than the perimeter, but greater than half the perimeter.

Let ABC (*Fig.* 2) be the triangle, O the given point. Then, by Nos. 60 and 61,

$$OA + OB < AC + BC \text{ but } > AB;$$
$$OB + OC < AB + AC \text{ but } > AC;$$
$$OA + OC < AB + BC \text{ but } > BC.$$

Add these inequalities, and divide by 2.

20. The sum of the diagonals of a quadrilateral is less than the sum and greater than half the sum of the sides.

21. Each side of a triangle is less than half the perimeter.

22. The altitude of an equilateral triangle is greater than half of one side.

23. Each altitude of a triangle is less than half the sum of the adjacent sides.

24. The sum of the three altitudes of a triangle is less than the perimeter. (Ex. 23.)

25. Each median of a triangle is less than half the sum of the adjacent sides.

Let ABC be the triangle, AM the median; produce AM to D, making $MD = AM$, join CD, and show that $CD = AB$.

26. The sum of the three medians of a triangle is less than the perimeter. (Ex. 25.)

27. If in a right triangle one of the acute angles is equal to twice the other, the hypotenuse is equal to twice the smaller leg. What are the values of the acute angles?

$\triangle ABC$, right angle at C, BC the smaller leg. Construct $\angle CAD = \angle CAB$, and produce BC to D.

28. Prove the converse of the preceding theorem.

29. The bisectors of the angles of a triangle meet in a point equidistant from the sides.

30. The perpendiculars erected at the middle points of the sides of a triangle meet in a point equidistant from the three vertices.

31. Lines drawn through the vertices of a triangle, parallel respectively to the opposite sides, form another triangle, the sides of which are bisected by the vertices of the first triangle. Compare the triangles as to magnitude.

32. The three altitudes of a triangle meet in one point.

33. The centre of a parallelogram is the middle point of every line drawn through the centre and limited by the sides.

34. Every line through the centre of a parallelogram divides its perimeter into two equal parts.

35. The line which bisects two sides of a triangle is parallel to the third side, and equal to half the third side.

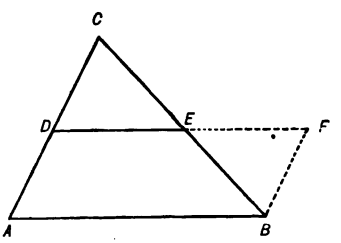


Fig. 3.

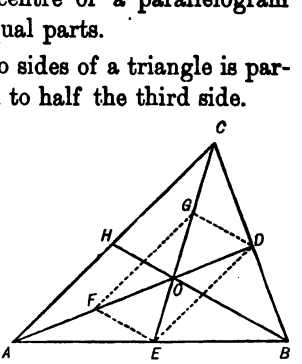


Fig. 4.

Produce (*Fig.* 3) DE, making $EF = DE$, and join BF. $\triangle DCE$ and BEF are equal, and $ABDF$ is a parallelogram.

36. A line parallel to the base of a triangle, and bisecting one side, bisects the other side, and the part intercepted between the two sides is equal to half the base.

Through D (*Fig.* 3) draw a line parallel to BC.

37. The perpendiculars dropped from the middle points of two sides of a triangle to the third side are equal. (Ex. 36.)

38. The lines which join the middle points of the sides of a triangle divide the triangle into four equal triangles.

39. The middle points of the sides of a rectangle are the vertices of a rhombus.

Draw the diagonals, and apply Ex. 35.

40. The middle points of the sides of a rhombus are the vertices of a rectangle.

41. If lines are drawn through the middle points of the sides of a square taken in order, what kind of a figure will they enclose? Prove.

42. The lines which join the middle points of the sides of any quadrilateral taken in order enclose a parallelogram, and the perimeter of the parallelogram is equal to the sum of the diagonals of the quadrilateral. What kind of parallelogram will be formed if the quadrilateral is a rectangle? a rhombus? a square?

43. The three medians of a triangle meet in a point which divides each median into two parts such that one is equal to twice the other.

In $\triangle ABC$ (*Fig.* 4) the medians are AD, CE, and BH. Let AD and CE meet in O, and F and G be the middle points of AO and CO respectively. $DEFG$ is a parallelogram, and $CO = 2\,OE$. Show in a similar way that BH must also pass through O.

44. The lines drawn through two opposite vertices of a parallelogram, to the middle points of the opposite sides, divide one of the diagonals into three equal parts.

Show that these two lines are parallel, and draw through one of the middle points a line parallel to the diagonal.

45. The middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

46. If the diagonals of a parallelogram are equal, the figure is a rectangle.

47. If the diagonals of a parallelogram are perpendicular to each other, the figure is either a rhombus or a square. When is it a rhombus? when a square?

48. The bisectors of the angles of a rhomboid enclose a rectangle.

49. The bisectors of the angles of a quadrilateral enclose another quadrilateral the opposite angles of which are supplementary. What kind of quadrilateral will be formed if the given quadrilateral is a parallelogram? a rectangle?

50. Lines drawn through the vertices of a quadrilateral, parallel to the diagonals, form a parallelogram twice as large as the quadrilateral.

51. The diagonals of an isosceles trapezoid are equal.

52. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

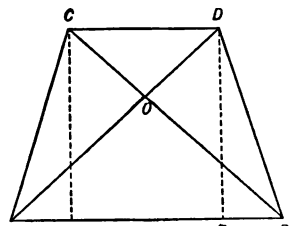


Fig. 5.

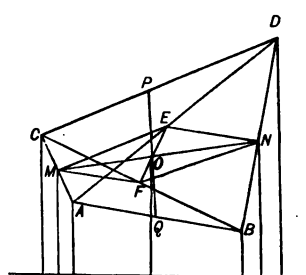


Fig. 6

Draw (*Fig.* 5) CE and $DF \perp CD$. Show that $\triangle ADF$ and BCE are equal, that $\triangle COD$ and AOB are isosceles, and that $\triangle AOC$ and BOD are equal.

53. In an isosceles trapezoid the lines joining the middle points of the sides, taken in order, form a rhombus.

54. The line joining the middle points of the bases of an isosceles trapezoid passes through the intersection of the diagonals and also through the intersection of the two legs produced.

55. The median of a trapezoid passes through the middle points of the two diagonals.

56. The straight line joining the middle points of the diagonals of a trapezoid is equal to half the difference of the bases.

57. Two trapezoids are equal if their sides taken in order are equal, each to each.

Divide each figure into a triangle and a parallelogram, and show that the two triangles are equal, and also the two parallelograms.

58. In any quadrilateral,

(i.) the lines joining the middle points of the opposite sides mutually bisect each other;

(ii.) the intersection of these lines bisects the line which joins the middle points of the diagonals;

(iii.) the distance of this intersection from any fixed line in the plane of the figure is the arithmetical mean of the distances of the four vertices of the quadrilateral from the same fixed line.

(i.) Let $MNPQ$ (*Fig.* 6) be the middle points of the sides of the quadrilateral $ABCD$. The figure formed by joining $MPNQ$ in order is a parallelogram (Ex. 42); therefore the diagonals MN and PQ mutually bisect each other at their imtersection O.

(ii.) Let E, F be the middle points of AD, BC. Show that $MENF$ is a parallelogram (Ex. 35); then EF divides MN into two equal parts. Therefore EF passes through the point O.

(iii.) Let C_1D_1 be any fixed line. Drop perpendiculars to it from the points A, B, C, D, M, N, O. By No. 71,

$$OO_1 = \frac{MM_1 + NN_1}{2}, \quad MM_1 = \frac{AA_1 + CC_1}{2}, \quad NN_1 = \frac{BB_1 + DD_1}{2}.$$

Therefore, $$OO_1 = \frac{AA_1 + BB_1 + CC_1 + DD_1}{4}.$$

CHAPTER II.

THE CIRCLE.

§ 3. Questions and Numerical Exercises.

1. What part of a circle may be called either a segment or a sector?

2. How many chords having a given length can be drawn through a point within a circle?

3. What are the longest and the shortest chords that can be drawn through a given point within a circle?

4. Locate the middle points of all chords parallel to a given line.

5. Locate the middle points of all chords which have a given length.

6. How many circles passing through two given points can be described? Locate their centres. Describe the position of the smallest circle. What is its radius?

7. What figure is formed by two tangents drawn through an exterior point, and the radii drawn to the points of contact? What relation exists between the angle formed by the tangents and the angle formed by the radii?

8. What is the value of an angle standing upon a chord which subtends an arc of 150°? Also of an angle inscribed in a segment of 200°?

9. The difference of two inscribed angles is 4°; find the difference of the corresponding angles at the centre.

10. Two tangents drawn through an exterior point make an angle of 60°. Find the value of the arc contained between them.

11. Through the extremities of an arc of 45° tangents are drawn; find the angle made by them.

12. Find the angle formed by tangents drawn through a point, the distance of which from the centre is equal to the diameter of the circle.

13. If the tangents drawn through an exterior point are perpendicular to each other, what is the value of the arc contained between them?

14. If the angle between two tangents drawn through an exterior point is 60°, find the angle between the chord of contact and one of the tangents.

15. If a triangle ABC is inscribed in a circle, and its angles denoted by A, B, C, respectively, find the angle which the tangent through the vertex A makes with the side BC. Under what condition will this tangent be parallel to BC?

16. A circle and an angle are situated in the same plane. What is the measure of the angle,

(i.) if the vertex is anywhere within the circle?

(ii.) if the vertex is in the circumference?

(iii.) if the vertex is without the circle, and both sides of the angle cut the circle?

(iv.) if the vertex is without the circle, and both the sides touch the circle?

(v.) if the vertex is without the circle, and one side cuts while the other side touches the circle?

17. Describe the relative position of two circles if the line of centres,

(i.) is greater than the sum of the radii;

(ii.) is equal to the sum of the radii;

(iii.) is less than the sum but greater than the difference of the radii;

(iv.) is equal to the difference of the radii;

(v.) is less than the difference of the radii.

Illustrate each case by a figure.

18. Distinguish between *external* contact and *internal* contact of two circles. Which kind of contact only is possible in the case of two equal circles?

19. Two circles touch each other, and their centres are 8 inches apart. The radius of one of the circles is equal to 5 inches. What is the radius of the other? (Two solutions.)

20. If the radii of two concentric circles are denoted by a and b, respectively, find the radius of a third circle which shall touch both the given circles and contain the smaller.

§ 4. Theorems.

1. The radius which bisects an angle at the centre bisects the corresponding chord, and is perpendicular to it.

2. The radius which bisects a chord bisects also the corresponding arc, and is perpendicular to the chord.

3. The perpendicular erected at the extremity of a radius is a tangent to the circle.

4. The perpendicular dropped from the centre of a circle to a tangent passes through the point of contact.

5. The tangents drawn through the extremities of a diameter are parallel.

6. Two parallel lines intercept upon the circumference of a circle equal arcs.

7. In the same circle, or in equal circles, if the sum of two arcs is less than the circumference, the greater arc is subtended by the greater chord; and conversely.

8. If an angle, the sides of which pass through the extremities of a chord, is equal to the inscribed angle on the same side of the chord, the vertex of the angle lies in the circumference of the circle.

Show that, if the vertex were at a point without the circle, or at a point within the circle, we are led to a conclusion which contradicts the hypothesis.

9. If through any point in the convex arc included between two tangents a third tangent is drawn, a triangle will be formed, the perimeter of which is constant and equal to the sum of the two tangents.

10. If a circle is inscribed in a triangle ABC, and a, b, c, denote the sides BC, AC, AB, and p half the perimeter, the segments made by the points of contact and adjacent to the vertices A, B, C, respectively, are equal to $p-a$, $p-b$, and $p-c$.

11. The perimeter of an inscribed equilateral triangle is equal to half the perimeter of the circumscribed equilateral triangle.

12. The diameter of the circle circumscribed about a right triangle is equal to the hypotenuse.

13. The radius of the circle inscribed in an equilateral triangle is equal to one-third of the altitude of the triangle.

14. In a circumscribed quadrilateral the sum of two opposite sides is equal to the sum of the other two sides. (No. 89.)

15. A circle can be inscribed in a quadrilateral if the sum of two opposite sides is equal to the sum of the other two sides.

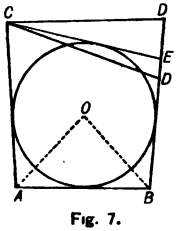


Fig. 7.

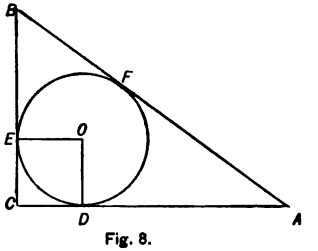


Fig. 8.

The bisectors of two angles A and B (*Fig.* 7) meet at a point O equidistant from the sides AC, AB, and BD (No. 69); therefore a circle can be described, with O as centre, which will touch these three sides. The fourth side CD must either (i.) cut this circle, or (ii.) lie wholly without it, or (iii.) touch it.

(i.) Suppose CD cuts the circle, and CE a tangent passing through C. Then, by the last theorem,

$$AC + BE = AB + CE$$

But by hypothesis, $AC + BD = AB + CD$

Therefore, $BE - BD = CE - CD$

or $DE = CE - CD$

which is impossible by No. 61.

(ii.) Suppose CD lies without the circle. Then, by subtracting the first of the above equalities from the second, we have

$$BD - BE = CD - CE,$$

or $$DE = CD - CE,$$

which is also impossible by No. 61. Therefore CD must touch the circle and coincide with CE.

16. In what kinds of parallelograms can a circle be inscribed? Prove.

17. The tangents drawn through the vertices of an inscribed rectangle enclose a rhombus.

18. The tangents drawn through the vertices of an inscribed quadrilateral, in which two opposite angles are right angles, form a trapezoid.

If the quadrilateral is a deltoid, prove that the trapezoid will be isosceles.

19. The diameter of the circle inscribed in a right triangle is equal to the difference between the sum of the legs and the hypotenuse.

Show first that $ODCE$ (*Fig.* 8) is a square. Then, $CD + CE =$ the diameter of the inscribed circle.

Now,
$$\begin{aligned} CD + CE &= AC + BC - (AD + BE) \\ &= AC + BC - (AF + BF) \\ &= AC + BC - AB. \end{aligned}$$

20. The angle formed by two tangents is equal to twice the angle between the chord of contact and the radius drawn to a point of contact.

21. If the tangents drawn from an exterior point to a circle form an angle of 120°, the distance of the point from the centre is equal to the sum of the tangents.

22. An isosceles trapezoid is inscriptible; that is, a circle can be circumscribed about it.

23. If in a circle two chords are drawn, and the middle point of the arc subtended by one chord is joined to the extremities of the other chord, the two triangles thus formed are mutually equiangular, and the quadrilateral thus formed is inscriptible.

(i.) The triangles have a common angle; and their other angles are equal because they have the same measure. (ii.) The opposite angles of the quadrilateral have for their measure a semi-circumference.

24. If A, B, C, A', B', C' are six points in a circumference, such that AB is parallel to $A'B'$, and AC parallel to $A'C'$, then BC' is parallel to $B'C$.

25. The bisectors EF and GH (*Fig.* 9.) of the angles formed by the opposite sides of an inscribed quadrilateral are perpendicular to each other.

By No. 105, $AF - CM = FB - MD$,
and $AH - BN = CH - DN$,
whence, $FH - CM - BN = FB + CH - MN$,
or $FH + MN = HM + FN$;
$\therefore \angle FOH = \angle HOM$ (No. 104);
$\therefore EF$ is $\perp$ to GH.

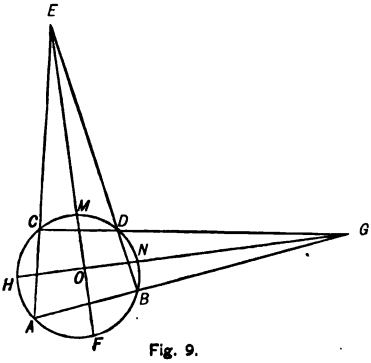


Fig. 9.

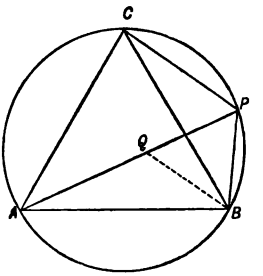


Fig. 10.

26. If a circle is circumscribed about an equilateral triangle, and any point in the circumference is joined to the three vertices, the greatest of the joining lines is equal to the sum of the other two lines.

Let ABC (*Fig.* 10) be the triangle, P the point. Upon AP take $PQ = PB$. Then $\angle QBP = \angle QPB$. But $QPB = 60°$, (No. 99); whence $\triangle PQB$ is equiangular, and therefore equilateral. Also, $\angle AQB = 120°$, (No. 32), and $\angle BPC = 120°$, (No. 99). In the $\triangle AQB$ and BPC, $BQ = BP$, $\angle AQB = \angle BPC$, $\angle QAB = \angle BCP$, (No. 99). $\therefore \triangle AQB = \triangle BPC$, and $AQ = CP$. Hence, $PQ + QA = PB + PC$.

27. Through a point A without a circle two secants are drawn: one, AOB, passing through the centre O; the other, ACD, so that the part AC without the circle is equal to the radius. Prove that the angle OAD is equal to one-third of the angle BOD.

28. If a circle is circumscribed about any triangle, the feet of the perpendiculars dropped from any point in the circumference to the sides of the triangle lie in one straight line.

The circle (*Fig.* 11) having AP for diameter passes through E and D (No. 107); $\therefore \angle APD = \angle AED$. Similarly, it follows that $\angle FPC = \angle FEC$. Hence, the $\triangle APC$ and DPF are equal; $\therefore \angle APD = \angle FPC$. Hence, $\angle AED = \angle FEC$; $\therefore$ DE and EF form one straight line.

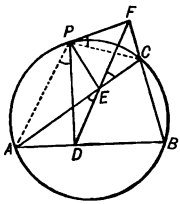


Fig. 11.

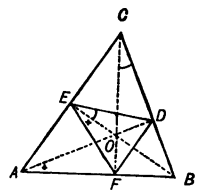


Fig. 12.

29. The altitudes AD, BE, CF, of the triangle ABC, bisect the angles of the triangle DEF.

The steps of the proof are as follows: the $\triangle OAF$, OEF (*Fig.* 12) are equal, because the figure $AFOE$ is inscriptible; the $\triangle OCD$, OED are equal because the figure $EODC$ is inscriptible; and the $\triangle OAF$, OCD are equal because the figure $AFDC$ is inscriptible. Therefore, $\angle OEF = \angle OED$.

30. Upon the three sides of a given triangle equilateral triangles are described outwardly. Prove that the three lines which join their vertices to the opposite vertices of the given triangle (i.) are equal; (ii.) cut each other at angles of 120°; (iii.) intersect in one point.

31. If a chord is divided into three equal parts, and radii are drawn through the points of division, will these radii divide the corresponding arc into equal parts? Prove.

32. Let A be any point of a diameter; B the extremity of a radius perpendicular to the diameter; P the point in which BA meets the circumference; C the point in which the tangent through P meets the diameter produced. Prove that $AC = PC$.

33. All chords of a circle which touch an interior concentric circle are equal, and are bisected at the points of contact.

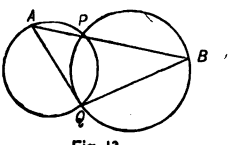


Fig. 13.

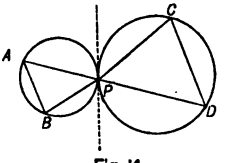


Fig. 14.

34. Two circles cut each other in the points P and Q, and through P the line APB is drawn, meeting the circles in A and B. Prove that the angle AQB is constant, whatever be the direction of APB.

It is sufficient to prove that the two angles A and B (*Fig.* 13) remain constant.

35. Two circles touch in the point P, and through this point two lines are drawn meeting one of the circles in A and B, the other in C and D. Prove that the chords AB and CD are parallel.

Draw a common tangent through P (*Fig.* 14), and compare the angles thus formed with the angles A and D, B and C.

36. Two circles touch in the point P, and APB is a line meeting the circles in A and B. Prove that the tangents through A and B are parallel.

37. If two circles touch internally, and the diameter of the smaller is equal to the radius of the larger, the circumference of the smaller bisects every chord of the larger which can be drawn through the point of contact.

38. Two circles touch internally in the point P, and AB is a chord of the larger circle touching the smaller in the point C. Prove that PC bisects the angle APB.

§ 5. Loci.

1. *Definitions.* There is an indefinite number of points in a plane so situated that they satisfy any *one* geometric condition. Points, however, *taken at random*, will not satisfy it. The points which satisfy it are always confined to a certain *line* or *group of lines*. This line or group of lines is called the **Locus** of the point which satisfies the given condition.

Since a line may be regarded as traced by a moving point, we may also define a locus as *the line generated by a point moving so that it always satisfies a given condition.*

The only loci which are considered in Elementary Plane Geometry are those which are either straight lines or the circumferences of circles.

2. *Scholium.* In order to prove *completely* that a certain line is the locus of a point which satisfies a given condition, two propositions must be established:

(i.) that every point in the line satisfies the given condition (the direct proposition);

(ii.) *either* that every point satisfying the given condition is in the line (the converse proposition), *or* that every point not in the line fails to satisfy the given condition (the contrary proposition).

3. The locus of a point which has a given distance from a given point is the circumference of a circle with the given point as centre and the given distance as radius.

4. The locus of a point equidistant from two given points is the perpendicular which bisects the line joining the points.

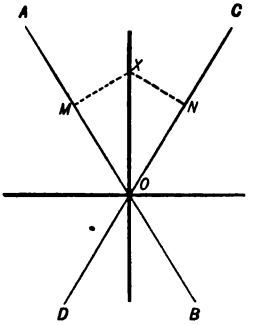


Fig. 15.

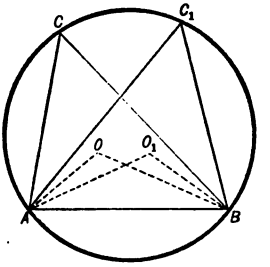


Fig 16.

5. The locus of a point equidistant from two given intersecting lines consists of the two bisectors of the angles formed by the given lines.

Let AB, CD (*Fig.* 15) be the given lines; O their intersection; X a point in one of the bisectors; MX and NX the distances from X to the lines. The right triangles OMX and ONX are equal; $\therefore MX = NX$.

Conversely, let X be a point such that its distances MX, NX from the given lines are equal, and prove that OX bisects the angle AOC.

6. The locus of a point equidistant from two given parallel lines is a line parallel to the given lines, and equidistant from them.

7. The locus of a point which is at a given distance from a given line consists of the two parallels to the given line, drawn at the given distance from it, one on each side of it.

8. The locus of the vertex of a right triangle, having a given hypotenuse as the base, is the circumference described upon the given hypotenuse as diameter.

9. The locus of the vertex of a triangle, having a given base and a given angle at the vertex, is the arc which forms with the base a segment capable of containing the given angle.

10. The locus of the centres of the circles inscribed in triangles, having a given base and a given angle at the vertex, is the arc which forms with the base a segment capable of containing an angle equal to 90° plus half the given angle at the vertex.

The bisectors of the angles CAB and CBA, C_1AB and C_1BA, etc., (*Fig.* 16), determine by their intersections the centres O, O_1, etc., of the inscribed circles. Then show that the $\measuredangle AOB$, AO_1B are constant, and each equal to $90° + \frac{1}{2} ACB$.

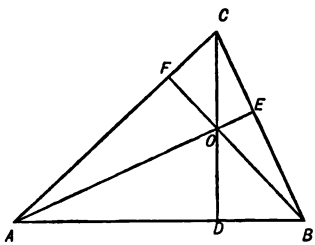


Fig. 17.

11. The locus of the intersections of the altitudes of triangles, having a given base and a given angle at the vertex, is the arc forming with the base a segment capable of containing an angle equal to the supplement of the given angle at the vertex.

Let ABC (*Fig.* 17) be one of the triangles, O the intersection of the altitudes. Show that in the quadrilateral $EOFC$ the $\angle EOF = 180° - \angle ACB$. Therefore, the $\angle AOB = 180° - \angle ACB$, and is constant.

12. The locus of the middle points of all chords in a given circle, which have a given length, is the concentric circumference having for radius the distance from one of the chords to the centre.

13. In a given circle having O as centre, the locus of the middle points of chords which pass through a given point P is the arc of the circle having OP for diameter, and contained within the given circle.

Three cases are to be considered: (i.) P in the given circumference; (ii.) P within the given circle; (iii.) P without the given circle.

14. The locus of the extremities of tangents to a given circle, which have a given length, is the circumference of a concentric circle, the radius of which is the distance from the centre to the extremity of one of the tangents.

15. The locus of points, from which tangents drawn to a given circle form a given angle, is the circumference of a concentric circle having for radius the distance from the centre to the intersection of any two tangents which form the given angle.

Construct an angle at the centre equal to the supplement of the given angle, and draw tangents through the extremities of the radii which form its sides; these tangents meet in one point of the required locus.

16. Find the locus of a point which is at a given distance from a given circumference.

17. Find the locus of the vertices of triangles having a given base and a given altitude.

18. Find the locus of the middle points of lines drawn from a given point P to a given line L.

19. Find the locus of the extremities of lines meeting in a given point P and bisected by a given line L.

20. Find the locus of points the sum of whose distances from two given parallel lines is equal to a given length.

Let d = distance between the given lines, l = the given length. If $l > d$, the locus consists of the two lines parallel to the given lines, and lying without them, each at the distance $\frac{1}{2}(l - d)$ from the nearer line. If $l = d$, every point between the two lines satisfies the condition. If $l < d$, there is no locus.

21. Find the locus of points the difference of whose distances from two given parallel lines is equal to a given length.

22. Find the locus of points the sum of whose distances from two given intersecting lines is equal to a given length.

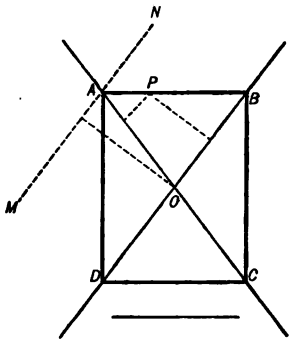


Fig. 18.

Let the given lines AC, BD, meet in O (*Fig.* 18). Draw $MN \parallel$ to BD, and at a distance from it equal to the given length, and let MN meet AC in A. With O as centre, and OA as radius, cut the given lines in B, C, and D. Then show that the figure $ABCD$ is a rectangle, and that its sides form the required locus.

23. Find the locus of points the difference of whose distances from two given intersecting lines is equal to a given length.

The locus consists of the prolongations of the sides of the rectangle $ABCD$ (*Fig.* 18).

24. An angle moves so that its magnitude remains constant and its sides pass through two fixed points; find the locus of the vertex.

25. A straight line moves so that it remains parallel to a given line, and one end touches a given circle; find the locus of the other end.

26. A ladder rests with one end against a vertical wall, and the other end upon a horizontal floor. If the ladder fall by sliding along the floor, find the locus of its middle point.

27. If ABC is an equilateral triangle, find the locus of a point P such that $PA = PB + PC$. (§ 4, Ex. 24).

28. Given two points, P, Q, and a straight line through Q. Find the locus of the foot of the perpendicular from P to the given line as the latter revolves around Q.

29. AB is a fixed diameter of a circle, and the chord AC is produced to M, so that $MC = BC$. Find the locus of M as AC turns about A.

§ 6. Problems. General Remarks.

1. A geometric problem requires the construction of a point, line, or figure, which shall satisfy given conditions.

Problems in Elementary Geometry are restricted to those which can be solved with the aid of ruler and compasses.

2. A problem is said to be **determinate,** if it has one, two, or any finite number of solutions; **indeterminate,** if it has an infinite number of solutions; **impossible,** if it has no solution; and **over-determinate,** if more conditions than are necessary for the solution are given.

3. A problem is often possible for certain values of the given magnitudes or certain positions of the given points, and impossible for other values or other positions.

4. Over-determinate problems are impossible, except in some cases for particular values of the given magnitudes, or particular positions of the given points.

To construct a right triangle, having given the two legs, or to draw a tangent to a given circle through a given point without the circle, are examples of determinate problems. To find a point equidistant from two given points, or to construct a parallelogram, having given two adjacent sides, are examples of indeterminate problems. The problem, "to draw a tangent through a given point within a given circle," is always impossible. The problem, "to construct an isosceles triangle, having given the base and one leg," is impossible if the given leg is less than half the given base. The problem, "to describe a circumference through four given points," is over-determinate, and is impossible except when the points are so placed that the opposite angles of the quadrilateral formed by joining the points are supplementary (No. 107).

5. The systematic solution of a problem consists of four parts: the **Analysis,** or course of thought by which the construction of the required figure is discovered; the **Construction** of the required figure; the **Proof** that the required figure satisfies all the given conditions; the **Discussion** in which we determine the number of solutions for different values of the given magnitudes, or different positions of the given points.

6. Problems differ so widely in their nature that no general rule can be given by means of which the analysis of any problem may be effected. In all cases, however, the first step in the analysis of a problem is to construct a figure representing the problem as already solved, in order to examine more easily the relations of the various parts. The analysis is to be considered finished when it has been shown how the required figure may be constructed by means of the elementary problems, or by means of any other problems which have been already solved.

7. In general, the most instructive as well as the most difficult part of the solution is the analysis; and after a few problems have been completely and systematically worked out, it is advisable to consider a problem finished when its analysis has been effected, and the number of solutions noted, and to devote the time saved by omitting the construction and proof to the study of new problems.

§ 7. Problems. Construction of Points.

1. *Number of given Conditions.* *Two* given conditions are necessary to determine the position of a point.

2. *Notation.* P, P_1, P_2, etc., denote given points; L, L_1, L_2, etc., denote given straight lines.

Method of Analysis. The two conditions which the point must satisfy are considered separately; to each of them will correspond a certain locus for the point. If these loci are known, or can be found, the problem is solved; for, since the point is in both loci, it must be their point of intersection. This is the *Method of Loci.*

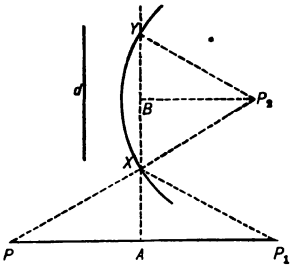


Fig. 19.

3. To find a point X which shall be equidistant from P and P_1, and at a given distance d from P_2.

Analysis. (*Fig.* 19.) Let the point X satisfy the given conditions. Since X is equidistant from P and P_1, one locus in which it must lie

is the perpendicular bisecting PP_1. And since X is at the distance d from P_2, another locus is the circumference having P_2 as centre and d as radius. Therefore X is the intersection of these two loci.

Construction. Join PP_1, erect a perpendicular at A, the middle point of PP_1, and describe a circumference with P_2 as centre and d as radius. The points X, Y, in which this circumference cuts the perpendicular, satisfy the given conditions.

Proof. Join X to P, P_1, and P_2. $P_2X = d$ (const.). $\triangle P_1AX = \triangle PAX$ (No. 40); $\therefore P_1X = PX$.

Discussion. The problem has two solutions, one solution, or no solution, according as the circumference cuts the perpendicular, touches it, or does not meet it at all; that is (if P_2B is $\perp$ to AY), according as $d > P_2B$, $d = P_2B$, or $d < P_2B$.

In L to find a point X so that:

4. X shall be at the distance d from P.

5. X shall be equidistant from P and P_1.

6. X shall be at the distance d from L_1.

7. X shall be equidistant from two given parallel lines.

To find a point X such that:

8. X shall be at the distance a from P, and the distance b from P_1.

9. X shall be at the distance a from P, and at the distance b from L.

10. X shall be equidistant from two parallel lines, L and L_1, and at the distance a from P.

11. X shall be equidistant from two intersecting lines, L and L_1, and at the distance a from P.

12. X shall be equidistant from P and P_1, and also equidistant from two parallel lines, L and L_1.

13. X shall be equidistant from P and P_1, and also equidistant from two intersecting lines, L and L_1.

14. X shall be equidistant from two given parallel lines, and also equidistant from two given intersecting lines.

15. X shall be equidistant from three given points.

16. X shall be equidistant from three given lines, two of which are parallel. (Two solutions.)

17. X shall be equidistant from three given intersecting lines. (Four solutions.)

18. To find a point X in one side of a given triangle which shall be equidistant from the other two sides.

19. To find a point X equidistant from two adjacent sides of a given quadrilateral, and also equidistant from the other two sides.

20. In a given line to find a point from which the lines drawn to two given points will be perpendicular to each other.

21. Given four points, P_1, P_2, P_3, and P_4; to find a point X such that the angles P_1XP_2 and P_3XP_4 shall each be equal to 90°.

22. To find a point from which the three sides of a triangle are seen under equal angles.

This problem will be impossible if one of the angles of the triangle is greater than 120°. Why?

23. Two villages are situated on opposite sides of a river. Show by construction how to locate a bridge so that it shall be equidistant from the two villages. Also solve the same problem for the case in which the villages are on the same side of the river.

24. A man desires to build a house so that it shall be two miles from a certain railway station, and also just half way between two straight railroads which meet at the station. Find by construction the proper location for the house. How many solutions are there?

25. A straight railway passes two miles from a certain town. A place is described as four miles from the town and one mile from the railway. Find by construction how many places answer the description.

26. The positions A, B, C of three church-spires being marked on a map, it is required to locate on the map the position M of a house when the angles AMB, BMC, formed by horizontal lines drawn from the house to the three spires, have been measured on the ground.

§ 8. Problems. Construction of Circles.

1. *Number of given Conditions.* *Two* given conditions determine a locus for the centre. *Three* given conditions completely determine the position and size of the circle.

2. *Notation.* K, K_1, K_2, etc., denote circles; O, O_1, O_2, etc., denote their centres; r, r_1, r_2, etc., denote their radii.

3. *Method of Analysis.* The method of loci is always applicable; and in a few cases auxiliary points and lines are required.

For the sake of brevity the expression "in K" will be used, with the meaning "in the circumference of K"; also, in future, when ambiguity of meaning is impossible, instead of "circumference" the word "circle" will often be substituted.

To find the locus of the centre of a circle which has a given radius r, and also:

4. Passes through P.

5. Touches L.

6. Touches K externally.

7. Touches K internally.

8. Cuts L under the chord a.

9. Cuts K under the chord a.

10. Cuts K in a diameter.

11. Cuts K at right angles.

To find the locus of the centre of a circle which:

12. Passes through P and P_1.

13. Touches L in the point P.

14. Touches K in the point P.

15. Touches two intersecting lines L and L_1.

16. Touches two parallel lines L and L_1.

17. Touches two concentric circles K and K_1.

In the last two cases the two given conditions determine the radius as well as one locus for the centre.

To construct a circle, with the radius r and the centre in L, which shall:

18. Pass through P. 19. Touch K.

To construct a circle, with the radius r and the centre in K, which shall:

20. Pass through P. 21. Touch L.

To construct a circle, with the radius r, which shall:

22. Pass through P and P_1.

23. Pass through P and touch L.

24. Pass through P and touch K.

25. Pass through P and cut K in a diameter.

26. Pass through P and cut K at right angles.

27. Pass through P and cut from L a chord equal to a.

28. Touch L and L_1.

29. Touch L and K.

30. Touch K and K_1.

31. Touch L and cut K at right angles.

32. Touch L and cut K in a diameter.

33. Touch K and cut K_1 in a diameter.

34. Cut from L and L_1 chords equal to a and a_1.

35. Cut K and K_1 in diameters.

36. Cut both K and K_1 at right angles.

37. Cut K in a diameter and K_1 at right angles.

To construct a circle with its centre in L which shall:

38. Pass through P and P_1.

39. Touch L_1 at P.

40. Touch K at P.

41. Touch L_1 and L_2.

42. Touch two concentric circles, K and K_1.

To construct a circle, with its centre in K, which shall:

43. Pass through P and P_1.

44. Touch L and L_1.

45. Touch L at P.

46. Touch K_1 at P.

47. Touch two concentric circles K_1 and K_2.

To construct a circle which shall satisfy the conditions in Ex. 16, and also:

48. Pass through P.

49. Touch L_2.

50. Have its centre in L_2.

51. Have its centre in K.

52. Touch K externally.

53. Touch K internally.

54. Cut K in a diameter.

55. Cut K at right angles.

To construct a circle which shall satisfy the conditions in Ex. 17, and also:

56. Pass through P.

57. Touch L.

58. Have its centre in L.

59. Have its centre in K_2.

60. Touch K_2 externally.

61. Touch K_2 internally.

62. Cut K_2 in a diameter.

63. Cut K_2 at right angles.

64. To construct a circle which shall touch L in P, and pass through P_1.

65. To construct a circle which shall touch K in P, and pass through P_1.

66. To construct a circle which shall touch three given intersecting lines, L, L_1, and L_2.

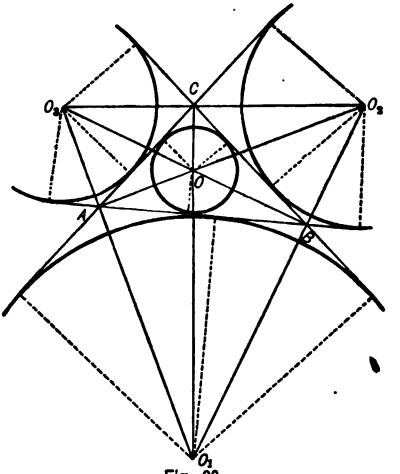


Fig. 20.

Analysis, by means of § 5, Ex. 5. (*Fig.* 20.) By drawing the bisectors of the exterior angles of the triangle ABC, four solutions in all are obtained. The circle having its centre within the triangle is the inscribed circle. The other circles are called the *escribed* circles. If the given lines meet in a point, there is no solution.

67. To construct a circle which shall touch L in P, and also touch L_1.

Analysis. Produce L and L_1 till they meet.

68. To construct a circle which shall touch L, and also touch K in P.

Analysis. Draw a tangent at P, and the problem is reduced to Ex. 67.

69. To construct a circle which shall touch L in P, and also touch K.

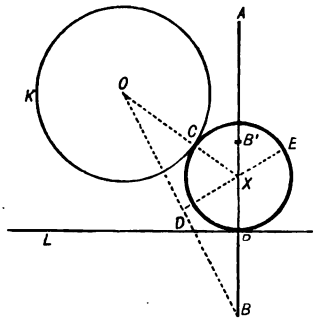


Fig. 21.

Analysis. (*Fig.* 21.) Suppose the problem solved, and let X be the centre of the required circle. One locus for X is the perpendicular PA erected at P. If AP is produced to B, so that PB is equal to the radius OC of the given circle, then $XB = XO$; therefore X must also lie in the perpendicular erected at the middle point of the line OB.

Another solution is obtained by taking the point B' in PA so that $PB' = OC$, joining OB', and erecting a perpendicular at the middle point of OB'.

To construct a circle with its centre in L, which shall:

70. Cut L in P and touch L_1.

71. Cut L in P and touch K.

Analysis of Ex. 70. Suppose the problem solved, X the centre of the required circle, Q the point of contact with L_1. Draw $PA \perp$ to L_1, and show that $\angle XPQ = \angle APQ$.

Analysis of Ex. 71. Similar to that of Ex. 69.

§ 9. Problems. Construction of Straight Lines.

1. *Number of given Conditions.* *Two* conditions are necessary to determine a straight line.

2. *Notation.* L, L_1, L_2, etc., denote straight lines given in position; a and b denote given lengths.

3. *Method of Analysis.* If one point in the required line is known, we should try to determine another point by the method of loci. If no point in the line is known, or if the method of loci cannot be directly applied, we must seek for auxiliary points or lines which have known relations to the required line, and which can be determined by means of loci, or known theorems; in this way the problem can be reduced to problems already solved, or easily solved, by means of loci or known theorems.

Through a given point in a given line to draw a line making with the given line, an angle of:

4. 60°. 6. 120°. 8. 45°. 10. 75°.

5. 30°. 7. 150°. 9. 15°. 11. 135°.

12. To divide a right angle into three equal parts.

13. Given the sum and the difference of two lines, to construct the lines.

Analysis. Let a and b denote the given sum and the given difference, x and y the required lines. Then $x = \frac{1}{2}(a + b)$, $y = \frac{1}{2}(a - b)$.

14. To erect a perpendicular at the end of a given line without producing the line.

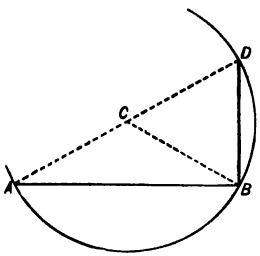


Fig. 22.

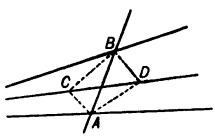


Fig. 23.

Analysis. (*Fig.* 22.) Let AB be the given line, and C any point equidistant from A and B. With C as centre, and CA as radius, describe a semicircle, and produce AC to meet it in D. Then DB is $\perp$ to AB.

15. To bisect the angle formed by two lines, without producing them to their point of intersection.

Analysis. (*Fig.* 23.) Draw any secant cutting the lines in A and B; the four bisectors of the angles at A and B meet in two points C, D, which are in the bisector required.

16. Explain how the distance between two objects on the ground, separated by an inaccessible swamp, can be found.

17. Explain how the distance from an accessible object to an inaccessible object can be found.

18. Given $L \parallel$ to L_1, and a point P; through P to draw a line cutting L in A, and L_1 in B, so that $AB = a$. When are there two solutions? one solution? no solution?

19. To draw through the vertex of a given angle, and without the angle, a line which shall make equal angles with the sides of the given angle.

Given P, L, and an angle with its vertex at A; to draw a line meeting the sides of the angle in B, C, which shall satisfy the following conditions:

20. $BC = a$, and $BC \parallel$ to L.

21. $BC = a$, and $AB = AC$.

22. BC passes through P, and $AB = AC$.

23. BC passes through P, and $PB = PC$.

24. BC passes through P, and $BC = 2\,AC$.

25. BC passes through P, and $BC = BA$.

Analysis of Ex. 20. Suppose the problem solved; let AB cut L in D, and a parallel to AB through C cut L in E; then $BCDE$ is a parallelogram.

Analysis of Exs. 21, 22. The bisector of the angle is perpendicular to the required line.

Discuss Exs. 22–25 for: (i.) P within the angle; (ii.) P without the angle; (iii.) P in one side of the angle.

Given a triangle ABC; to draw a line meeting AC in D, BC in E, which shall satisfy the following conditions:

26. $DE \parallel$ to AB, and $DE = BE$.

27. $DE \parallel$ to AB, and $CD = BE$.

28. $DE \parallel$ to AB, and $DE = AD + BE$.

29. $DE = a$, and $CD = BE$.

30. $DE = CD = BE$.

Analysis of Ex. 27. Suppose the problem solved, and parallels to AC drawn through B and E; this reduces the problem to 23.

Analysis of Ex. 29. Draw a parallel to DE through C, and a parallel to AC through E.

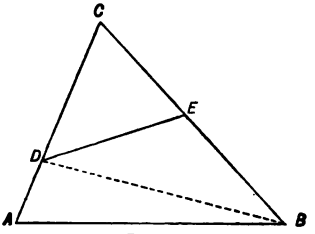


Fig. 24.

Analysis of Ex. 30. Suppose the problem solved, and draw BD (*Fig.* 24). The triangles BDE and DCE are isosceles; whence, $\angle DBE = BDE = \frac{1}{2} DEC = \frac{1}{2} DCE$, which is known. This determines the point D, and then E is easily found, since $DE = DC$.

Examine this problem for the special cases when $\angle ACE = 90°$, and when $\angle ACE = 120°$.

31. To draw a line which shall be equidistant from two given points. How many solutions are there?

32. Given P and P_1; to draw a line through P so that its distance from P_1 shall be equal to a.

33. Given P and P_1; to draw a line so that the distances from P and P_1, respectively, to the line, shall be a and b.

34. Given P, P_1, P_2; through P to draw a line which shall be equidistant from P_1 and P_2.

The line passing through P, and the middle point of P_1P_2, is one solution; and the line through P, parallel to P_1P_2, is another solution.

35. Given P, P_1, P_2; through P to draw a line so that the distances from P to the feet of the perpendiculars dropped from P_1, P_2, to the line, shall be equal.

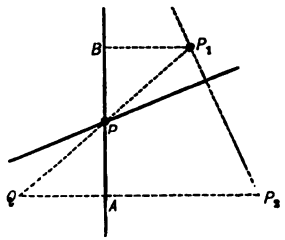


Fig. 25.

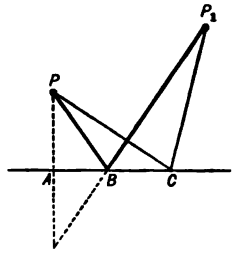


Fig. 26.

Analysis. The line through P, perpendicular to P_1P_2, is one solution. Produce P_1P (*Fig.* 25) to Q, making $PQ = PP_1$, and join QP_2. The line $APB \perp$ to QP_2 is another solution; for the triangles PBP_1 and PAQ are equal. $\therefore PB = PA$.

The point Q is called the *symmetrical* point of P_1, with respect to P.

36. Given L and two points, P, P_1; to draw lines through the points which shall meet in L, and make equal angles with L.

Analysis. The points may lie both on the same side of L, or one on each side of L. In each case there are two solutions. The line PP_1 forms one solution. The other solution is found by the use of symmetrical points. Draw (*Fig.* 26) $PA \perp$ to L; produce it, making $AQ = AP$, and join QP_1 meeting L in B; PB and P_1B are the required lines. The point Q is the *symmetrical* point of P, with respect to L.

37. To find the shortest path from P to P_1, which shall touch a given line L.

If the points are on opposite sides of L, join them; if on the same side (*Fig.* 26), make the same construction as in the last problem; PBP_1 is the path required.

The proof consists in showing that any other path which touches L, as $PC + CP_1$, is greater than $PB + BP_1$.

38. Given P and P_1 within the angle BAC; to find the shortest path from P to P_1, which shall touch both sides of the angle. (*Fig.* 27.)

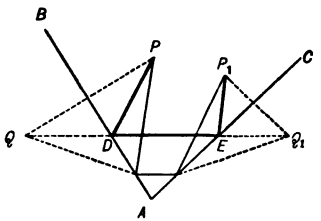


Fig. 27.

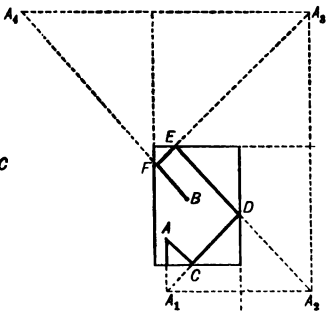


Fig. 28.

39. In a game of billiards two balls have the positions A, B (*Fig.* 28); to find by construction the path which the ball at A must pursue in order to hit the ball at B, after first striking the four sides of the table, assuming that the ball makes equal angles with the side of the table before and after impact.

Analysis. Suppose the problem solved, and $ACDEFB$ the required path. DC produced must pass through A_1, the symmetrical point of A with respect to the first side of the table; ED through A_2, the symmetrical point of A_1 with respect to the second side; etc. Therefore, construct A_1, A_2, A_3, A_4; then join BA_4, then FA_3, etc.

40. To construct upon a given line a segment capable of containing an angle of 30°. What relation exists between the length of the line and the radius of the circle?

41. Given K and L; to draw a line touching K and $\parallel$ to L.

42. Given K and L; to draw a line touching K and $\perp$ to L.

43. To draw a tangent through a given point in a given arc without making use of the centre.

Given K and P in K; to draw the following lines:

44. A chord through P at the distance a from O.

45. A diameter at the distance a from P.

46. A chord equal to b at the distance a from P.

Analysis for Exs. 44, 45. Describe a circle upon OP as diameter, and apply No. 102.

Analysis for Ex. 46. Apply Exs. 3, 12, of § 5, and Nos. 80, 87.

Given P within K; through P to draw the following lines:

47. The shortest chord.

48. A chord having a given length a.

Analysis for Ex. 48. Place in K any chord equal to a, and let C be its middle point; the required chord APB is tangent to the circle having O as centre and OC as radius. What are the maximum and minimum values of a for which the problem is possible?

Given P without K; through P to draw the following lines:

49. A secant cutting K in A and B so that $AB = a$.

50. A secant cutting K in A and B so that $PA = AB$.

Analysis for Ex. 49. See Analysis for Ex. 48.

Analysis for Ex. 50. Suppose PAB (*Fig.* 29) is the required line. Produce OA by its own length to C, and join PC. $\triangle ACP = \triangle AOB$; $\therefore PC = OA$; whence C can be determined.—**Second Analysis.** Draw $AD \perp$ to PB, meeting K in D. Then $DP = DB$, and DB is a diameter of K; whence B can be determined.

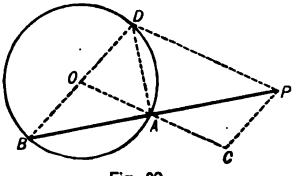


Fig. 29.

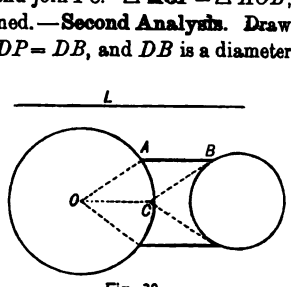


Fig. 30.

51. Given L, K, and K_1, exterior to one another; to place between K and K_1 a line parallel to L so that it shall have a given length a.

Analysis. Suppose the problem solved, and AB (*Fig.* 30) the required line. Draw OC parallel and equal to AB, and join OA, CB. The figure $OABC$ is a parallelogram, and $CB = OA$. From this the point B is easily determined.

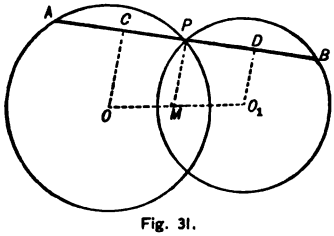


Fig. 31.

52. Given K and K_1, intersecting each other; to draw through one of the points of intersection a line so that the two intercepted chords shall be equal.

Analysis. (*Fig.* 31.) Let APB be the line required, $AP = PB$; C, D, and M, respectively, the middle points of AP, PB, and the line of centres OO_1. Join OC, MP, O_1D. OC and O_1D are perpendicular to AB, and parallel to each other; and, since $CP = PD$, MP is the median of the trapezoid $OCDO_1$. $\therefore MP$ is perpendicular to AB.

53. Given K and P, P_1 exterior to K; through P and P_1 to draw parallels cutting K so that the intercepted chords shall be equal.

54. Given two circles exterior to each other; to draw a common secant so that the intercepted chords shall have given lengths a, b.

Analysis. Inscribe in the given circles chords equal respectively to a and b, and describe circles concentric with the given circles and touching a and b. By § 5, Ex. 12, the problem is now reduced to No. 132.

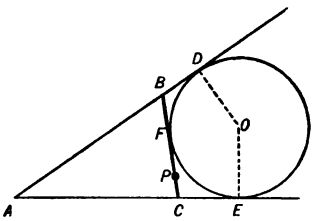


Fig. 32.

55. Given two intersecting circles; to draw through one of the points of intersection a common secant which shall have a given length.

56. Given two concentric circles K and K_1; through a given point P in K to draw a chord so that it shall be trisected by K_1.

57. Given an angle, and a point P within the angle; through P to draw a line which shall form, with the sides of the angle, a triangle having a perimeter equal to a given length a.

Analysis. (*Fig.* 32.) Let BPC be the required line. Construct the escribed circle touching BC in F, and the sides AB, AC produced in D, E. Since $BD = BF$, and $CE = CF$, $AB + BC + CA = AD + AE$. But $AD = AE$. $\therefore AD = AE = \frac{1}{2}a$. This determines the circle, and a tangent to the circle through P is the line required.

To construct a triangle, given :

12. c, h, γ. 15. u, v, γ. 18. $c, a, \angle bm$.

13. c, m, γ. 16. u, v, r. 19. $c, h, \angle bm$.

14. p, q, γ. 17. u, h, γ. 20. $c, \gamma, \angle bm$.

21. c, γ, and the condition that $a = c$.

22. c, and the altitudes upon the other two sides.

23. c, h, and the altitude upon a.

24. c, m, and the altitude upon a.

25. h, and the distances from the foot of h to the sides a and b.

B. Auxiliary triangles formed by drawing h, m, and t.

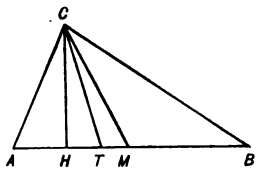


Fig. 34.

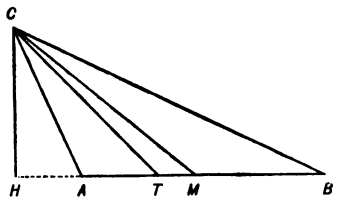


Fig. 35.

By drawing the altitude h two auxiliary right triangles ACH and BCH are formed; the required triangle ABC is equal to their sum if $\alpha < 90°$ (*Fig.* 34), and equal to their difference if $\alpha > 90°$ (*Fig.* 35). In the first case, $c = p + q$; in the second case, $c = p - q$.

The median m divides $\triangle ABC$ into the auxiliary $\triangle ACM$, BCM. It is also the hypotenuse of the auxiliary rt. $\triangle CHM$, in which one leg $= h$, and the other leg $= \frac{1}{2}c - q = \frac{1}{2}(p + q) - q = \frac{1}{2}(p - q)$.

The bisector t divides $\triangle ABC$ into the auxiliary $\triangle ACT$, BCT; and the $\angle tc = ATC = \beta + \frac{1}{2}\gamma = \beta + 90° - \frac{1}{2}(\alpha + \beta) = 90° - \frac{1}{2}(\alpha - \beta)$. Hence, if α and β are given, $\angle tc$ can be constructed. The bisector t is also the hypotenuse of the auxiliary rt. $\triangle CHT$, in which one leg $= h$, and the other leg $= p - u$.

4. *Method of Analysis.* In the simplest cases the fundamental problems (Nos. 112–133 of the Syllabus) can be immediately applied; in a few cases the loci in Exs. 8 and 9, § 5, are applicable; in most cases the construction must be found by the aid of other triangles, which can be more easily constructed from the given parts, and which are called *auxiliary triangles.*

It is convenient to subdivide the cases which will be here considered under the heads **A, B, C,** etc.

A. Cases in which the loci of **Ex.** 8 and 9 of § 5 can be applied.

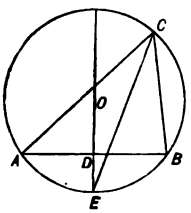


Fig. 33.

Let O (*Fig.* 33) be the centre of the circle circumscribed about the $\triangle ABC$. Draw $OE \perp$ to AB; then OE bisects AB in D, and the arc AB in E. Therefore, CE bisects the $\angle ACB$.

To construct a right triangle, given:

5. c, and the altitude upon c.

6. c, and one segment of c made by the altitude.

7. The two segments of c made by the altitude.

8. The two segments of c made by the bisector of the right angle.

9. c, and a line L in which C must lie.

10. c, and the distance from C to a line L.

11. c, and the distance from C to a point P.

Let $CD = 2\,CM$ (*Fig.* 36), and join AD, BD; then $ACBD$ is a parallelogram. (No. 56.) In the auxiliary $\triangle CBD$, $BD = b$, $CD = 2m$; $\angle CBD = 180° - \gamma$. Also, $\angle ABD = \alpha$, $\angle BAD = \beta$, $\angle CAD = \alpha + \beta$. If DE is perpendicular to BC, and DF perpendicular to AC, then DE and DF are equal respectively to the altitudes upon a and b of the $\triangle ABC$.

To construct a triangle, given :

83. a, b, m. **86.** m, h, γ. **89.** m, α, $\angle am$.

84. a, m, γ. **87.** m, a, α. **90.** m, $\angle am$, $\angle bm$.

85. m, α, β. **88.** a, m, $\angle bm$. **91.** a, $\angle am$, $\angle bm$.

92. m, h, and the altitude upon a.

93. m, α, and the altitude upon a.

94. m, and the altitudes upon a and b.

D. Auxiliary triangles employed when sums or differences of sides are given.

In $\triangle ABC$ (*Fig.* 37) let $BC > AC$. With C as centre and CA as radius, cut AB in F, BC in E, and BC produced in D, and draw AD, AE, CF. In the auxiliary $\triangle ABD$, $BD = a + b$, $\angle ADB = \frac{1}{2}\gamma$, $\angle BAD = \alpha + \frac{1}{2}\gamma = \alpha + \frac{1}{2}(180° - \alpha - \beta) = 90° + \frac{1}{2}(\alpha - \beta)$. In the auxiliary $\triangle ABE$, $BE = a - b$, $\angle AEB = \angle DAE + \angle ADE = 90° + \frac{1}{2}\gamma$, $\angle BAE = BAD - 90° = \frac{1}{2}(\alpha - \beta)$; also, $CF = b$, $\angle CFA = \alpha$. If $\triangle ABC$ is a rt. $\triangle$ (rt. $\angle$ at C), $\angle ADB = 45°$, $\angle AEB = 180° - 45° = 135°$.

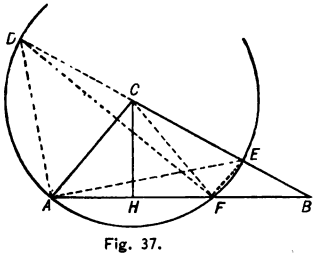


Fig. 37.

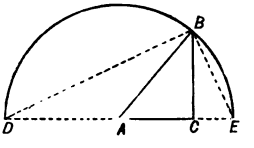


Fig. 38.

If, in a rt. $\triangle$ (*Fig.* 38), ABC, $c + b$, or $c - b$ is a given part, with A as centre and AB as radius, cut AC produced in D and in E, and draw BD, BE. In the rt. $\triangle BCD$, $CD = c + b$, $\angle BDC = \frac{1}{2}\alpha$; in the BCE, $CE = c - b$, $\angle BEC = 90° - \angle BDE = 90° - \frac{1}{2}\alpha$.

If the perimeter $a + b + c$ of a triangle is given, produce AB (*Fig.* 39), making $AD = AC$, $BE = BC$, and draw DC and EC. In the auxiliary $\triangle DCE$, $DE = a + b + c$, $\angle CDE = \frac{1}{2}\alpha$, $\angle CED = \frac{1}{2}\beta$. If $\triangle DCE$ can be constructed, it determines the required $\triangle ABC$, for these two triangles have the vertex C in common, and the points A, B are determined by erecting perpendiculars at the middle points of DC and EC.

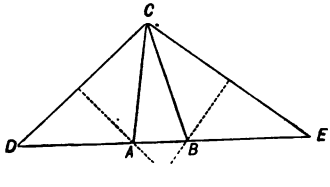


Fig. 39.

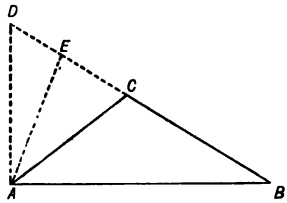


Fig. 40.

If $a + b - c$ is a given part, produce BC (*Fig.* 40), making $CD = CA$, and upon BD take $BE = BA$, and draw AD, AE. The $\triangle ADE$ is an auxiliary triangle in which $DE = a + b - c$, $\angle ADE = \angle DAC = \frac{1}{2}\gamma$, $\angle DEA = 180° - \angle AEB = 180° - \frac{1}{2}(180° - \beta) = 90° + \frac{1}{2}\beta$, $\angle DAE = 180° - 90° - \frac{1}{2}\beta - \frac{1}{2}\gamma = \frac{1}{2}(180° - \beta - \gamma) = \frac{1}{2}\alpha$.

To construct a triangle, given :

95. Rt. $\triangle$, $a + b$, c.

96. Rt. $\triangle$, $a + b$, β.

97. Rt. $\triangle$, $c + b$, a.

98. Rt. $\triangle$, $c + a$, β.

99. Isos. $\triangle$, $a + h$, c.

100. Isos. $\triangle$, $a + h$, γ.

101. Isos. rt. $\triangle$, $a + c$.

102. Equilat. $\triangle$, $a + h$.

103. $a + b$, β, γ.

104. $a + b$, h, α.

105. $a + b$, c, α.

106. $a + h$, c, β.

107. $a + b$, c, $\alpha - \beta$.

108. Isos. $\triangle$, $a + c$, γ.

109. Rt. $\triangle$, $a - b$, c.

110. Rt. $\triangle$, $a - b$, $\alpha - \beta$.

111. Rt. $\triangle$, $c - b$, a.

112. Rt. $\triangle$, $a - h$, β.

113. Isos. $\triangle$, $a - h$, c.

114. Isos. rt. $\triangle$, $a - h$.

115. Equilat. $\triangle$, $a - h$.

116. $a - b$, c, β.

117. $a-b$, α, β.

118. $a-b$, γ, $\alpha-\beta$.

119. $a-b$, p, β.

120. $a-b$, h, α.

121. $a-b$, c, α.

122. Isos. $\triangle$, $a-c$, α.

123. Rt. $\triangle$, $a+b+c$, α.

124. Isos. $\triangle$, $a+b+c$, γ.

125. $a+b+c$, α, β.

126. $a+b+c$, h, α.

127. $a+b+c$, h, γ.

128. $a+b+c$, q, α.

129. Rt. $\triangle$, $a+b-c$, α.

130. $a+b-c$, α, β.

E. Auxiliary triangles useful in certain cases in which $\alpha-\beta$ or $p-q$ is a given part.

In *Fig.* 37 draw the altitude CH of the $\triangle ABC$, and join DF, FE. In the $\triangle BCF$, $CF=b$, $BF=p-q$, $\angle BFC=180°-\alpha$, $\angle BCF=180°-\beta-(180°-\alpha)=\alpha-\beta$. In the $\triangle BDF$, $BD=a+b$, $\angle BDF=\frac{1}{2}\angle BCF=\frac{1}{2}(\alpha-\beta)$, $\angle BFD=180°-\beta-\frac{1}{2}(\alpha-\beta)=180°-\frac{1}{2}(\alpha+\beta)=180°-\frac{1}{2}(180°-\gamma)=90°+\frac{1}{2}\gamma$. In the $\triangle BEF$, $BE=a-b$, $\angle BEF=\angle EFD+\angle BDF=90°+\frac{1}{2}(\alpha-\beta)$, $\angle BFE=\angle BFD-\angle EFD=\frac{1}{2}\gamma$.

If, in the $\triangle ABC$, α is obtuse, F and H will lie in BA produced, and $BF=p+q$.

To construct a triangle, given :

131. a, b, $\alpha-\beta$.

132. a, h, $\alpha-\beta$.

133. p, q, $\alpha-\beta$.

134. $p-q$, a, b.

135. $p-q$, a, β.

136. $p-q$, α, β.

137. $p-q$, a, h.

138. $p-q$, a, $\alpha-\beta$.

139. $a+b$, $p-q$, β.

140. $a+b$, $p-q$, $\alpha-\beta$.

141. $a-b$, $p-q$, β.

142. Rt. $\triangle$, $p-q$, β.

143. Rt. $\triangle$, $p-q$, $a-b$.

144. Rt. $\triangle$, $p-q$, $\alpha-\beta$.

F. Auxiliary triangles useful when u or v, or both u and v, are among the given parts.

In the ΔABC (*Fig.* 41) upon CB take $CD = CA$, and join D to the foot T of the bisector CT. Also, draw $DE \parallel$ to CT. $\Delta DTC = \Delta ATC$. $\therefore DT = AT = v$. Also, $\angle TDE = \angle CTD = \angle ATC = \angle TED$. $\therefore TE = TD = v$. The ΔBDT has the parts $BD = a - b$, $BT = u$, $DT = v$, $\angle BDT = 180° - \alpha$, $\angle BTD = \alpha - \beta$; the ΔBED has the parts $BD = a - b$, $BE = u - v$, $\angle BDE = \frac{1}{2}\gamma$, $\angle BED = 180° - \beta - \frac{1}{2}\gamma = 90° + \frac{1}{2}(\alpha - \beta)$.

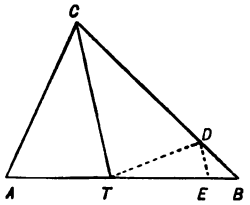


Fig. 41.

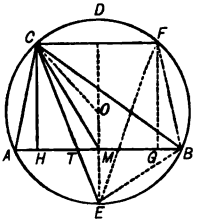


Fig. 42.

To construct a triangle, given:

145. u, v, α.	**149.** $u, v, a - b$.	**153.** $u, a - b, \alpha - \beta$.
146. u, v, β.	**150.** $u, v, \alpha - \beta$.	**154.** $a - b, u - v, \gamma$.
147. u, α, β.	**151.** $u, a - b, \alpha$.	**155.** $u - v, \beta, \gamma$.
148. v, α, γ.	**152.** $v, t, \alpha - \beta$.	**156.** $u - v, a - b, \alpha - \beta$.

G. Cases involving the circumscribed and the inscribed circles.

The centre O (*Fig.* 42) of the circle circumscribed about the ΔABC, is the intersection of the perpendiculars erected at the middle points of its sides. When the radius r of this circle is given, and also an angle of the triangle is either given or can be easily found, one side of the triangle can be constructed by means of No. 100. In some cases the loci of Ex. 8 and Ex. 9, § 5, are applicable. In other cases the following relations will be found useful:

Draw the altitude CH, the bisector CT, the median CM, the diameter $DOME$; this diameter bisects the arc AEB in E, therefore CT produced passes through E. Also, draw $CF \parallel$ to AB, $FG \perp$ to AB, and join BF, EF. $\Delta ACH = \Delta BFG$; therefore $\angle ABF = \alpha$, and

$\angle CBF = \alpha - \beta$. The figure $CHGF$ is a rectangle; therefore $CF = HG = BH - BG = BH - AH = p - q$. Also, $\angle HCT = \angle CED = \frac{1}{2}\angle CEF = \frac{1}{2}\angle CBF = \frac{1}{2}(\alpha - \beta)$.

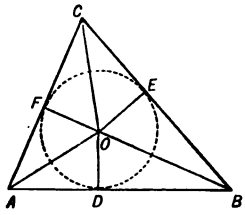


Fig. 43.

The centre O (*Fig.* 43) of the circle inscribed in the $\triangle ABC$, is the intersection of the bisectors AO, BO, CO, of the three angles. If perpendiculars OD, OE, OF are dropped from O to the sides, the $\triangle ABC$ will be divided into six auxiliary right triangles, in each of which one leg $= \rho$, and one of the acute angles = half of one of the angles of the $\triangle ABC$.

In problems in which ρ is a given part, the loci of Exs. 5, 7, and 15, of § 5, are sometimes applicable.

To construct a triangle, given r and also:

157. Rt. $\triangle$, α.	**167.** p, q.	**177.** u, v.
158. Rt. $\triangle$, $\alpha - \beta$.	**168.** α, β.	**178.** h, $p - q$.
159. Isos. $\triangle$, c.	**169.** h, γ.	**179.** m, $p - q$.
160. Isos. $\triangle$, α.	**170.** m, γ.	**180.** t, $p - q$.
161. Isos. $\triangle$, γ.	**171.** $a + b$, α.	**181.** c, $\alpha - \beta$.
162. Equilat. $\triangle$.	**172.** $a - b$, α.	**182.** h, $\alpha - \beta$.
163. a, b.	**173.** $a + b$, γ.	**183.** h, m.
164. c, α.	**174.** $a - b$, γ.	**184.** h, t.
165. c, h.	**175.** $a + b$, c.	**185.** $a - b$, $p - q$.
166. c, m.	**176.** $a - b$, c.	**186.** $a + b + c$, α.

To construct a triangle, given ρ and also :

187. Rt. $\triangle$, a.	**192.** Isos. $\triangle$, c.	**197.** h, a.
188. Rt. $\triangle$, a.	**193.** Isos. $\triangle$, γ.	**198.** h, γ.
189. Rt. $\triangle$, h.	**194.** Isos. $\triangle$, h.	**199.** a, h.
190. Rt. $\triangle$, t.	**195.** a, β.	**200.** p, h.
191. Equilat. $\triangle$.	**196.** c, a.	**201.** t, γ.

§ 11. Problems. Construction of Quadrilaterals.

1. *Number of Given Parts.* The trapezium is determined by *five* parts; the trapezoid by *four* parts; the rhomboid, the deltoid, and the isosceles trapezoid by *three* parts; the rectangle and the rhombus by *two* parts; and the square by *one* part.

2. *Notation.* $ABCD$ is any quadrilateral. $AB=a$, $BC=b$, $CD=c$, $DA=d$, $AC=f$, $BD=g$. The angles at A, B, C, D, respectively, are α, β, γ, δ. The diagonals f, g, intersect in E, and the $\angle AEB=\theta$. If a circle can be circumscribed, r is the radius; if a circle can be inscribed, ρ is the radius.

If the quadrilateral is a trapezoid, a and c are the bases, $a > c$, and h the altitude. If the quadrilateral is a parallelogram, h is the altitude upon AB.

In the rectangle, a and b are the sides; in the rhombus, α is the acute angle, β the obtuse angle; in the rhombus and the square, a is the side.

3. *Method of Analysis.* In general, problems of this kind are reducible to the construction of triangles by drawing one or both diagonals. The triangles require to be constructed only so far as may be necessary to determine the unknown vertices of the quadrilateral. The position of an unknown vertex can often be found by direct application of the Method of Loci. In some cases, especially in the

construction of trapezoids, certain auxiliary lines or figures are required. In constructing chord and tangent quadrilaterals, No. 89, and Exs. 3 and 6 of § 4, are to be kept in mind.

A. Construction of parallelograms.

To construct a square, given :

4. The side. 6. The diagonal. 7. $f+a$.
5. The perimeter. 8. $f-a$.

To construct a rectangle, given :

9. a, b. 13. $a, \angle af$. 17. $a, f+b$.
10. a, f. 14. $a, a+b$. 18. $a, f-b$.
11. a, θ. 15. $a+b, f$. 19. $a+b, \angle af$.
12. f, θ. 16. $a-b, f$. 20. $f-b, \angle af$.

To construct a rhombus, given :

21. f, g. 25. a, f. 29. $a+h$, a.
22. a, h. 26. f, a. 30. $f+g$, a.
23. h, a. 27. $a, f+g$. 31. $f+a$, a.
24. h, f. 28. $a, f-g$. 32. $f-h$, a.

To construct a rhomboid, given :

33. a, b, a. 41. a, f, g. 49. $a, f+g, \theta$.
34. a, f, a. 42. f, g, θ. 50. $a, f-g, \theta$.
35. a, b, f. 43. $f, g, \angle bg$. 51. $a+b, f$, a.
36. a, b, h. 44. $a, \theta, \angle ag$. 52. $a-b, f$, a.
37. a, g, h. 45. f, g, a. 53. $f+a, b$, a.
38. f, h, a. 46. g, a, θ. 54. $f-a, b$, a.
39. b, f, h. 47. a, h, θ. 55. $b+h, a$, a.
40. h, g, θ. 48. $f, h, \angle bg$. 56. $b-h, f$, a.

B. Construction of trapezoids.

The relations given by the following constructions are often useful.

Let $ABCD$ (*Fig.* 44) be any trapezoid. Draw CF and $BH \parallel$ to AD, CG and $AH \parallel$ to BD. Join H (the intersection of AH and BH) with F, C, G. The figures $ACGH$, $ADBH$, $FCBH$, are parallelograms. $BF = a - c$, $AG = AB + BG = a + c$, $\angle ACG = \theta$, $\angle CAH = \angle CGH = 180° - \theta$, $\angle CFH = \angle CBH = \alpha + \beta$, $\angle FCB = \angle FHB = 180° - (\alpha + \beta)$.

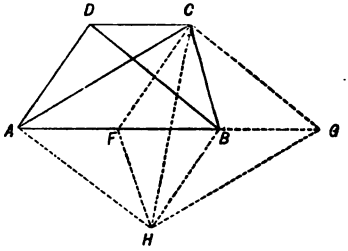


Fig. 44.

To construct an isosceles trapezoid, given :

57. a, b, f.	60. f, h, α.	63. $a+b, f, \alpha$.
58. a, b, h.	61. b, h, f.	64. $a-b, f, \alpha$.
59. a, f, α.	62. a, h, α.	65. $b+h, a, \alpha$.

To construct a trapezoid, given :

66. a, b, α, β.	76. a, f, g, θ.	86. $b, d, \alpha+\beta, \theta$.
67. a, b, d, α.	77. b, f, g, θ.	87. $a-c, \alpha, \beta, \theta$.
68. a, f, g, α.	78. f, g, α, θ.	88. $a-c, b, d, \theta$.
69. a, b, c, f.	79. a, c, f, θ.	89. $a+b, c, d, f$.
70. a, b, h, α.	80. a, b, c, θ.	90. $a-b, c, f, \alpha$.
71. a, b, β, θ.	81. b, d, α, θ.	91. $a+b, f, g, \beta$.
72. a, c, α, β.	82. b, α, β, θ.	92. $a-b, h, \alpha, \beta$.
73. a, b, c, d.	83. h, α, β, θ.	93. $a+b, c, d, \beta$.
74. a, b, c, α.	84. b, d, h, θ.	94. $a+c, b, d, \alpha$.
75. a, c, f, g.	85. b, h, α, θ.	95. $a-b, f, g, \theta$.

C. Construction of chord quadrilaterals.

In the cases in which r is not given, it is easy to construct a triangle, such that the circle circumscribed about this triangle will also be circumscribed about the required quadrilateral.

To construct a chord quadrilateral given :

96. $r, a, b, c.$	104. $r, a+b, c, d.$	112. $a, b, c, \alpha.$
97. $r, a, c, f.$	105. $r, a-b, f, g.$	113. $a, c, f, \beta.$
98. $r, a, f, g.$	106. $r, a+b, \alpha, \beta.$	114. $c, \beta, \gamma, \theta.$
99. $r, a, \gamma, \delta.$	107. $r, a-b, f, \theta.$	115. $f, g, \alpha, \theta.$
100. $r, a, b, \theta.$	108. $a, b, c, \beta.$	116. $a+b, c, f, \beta.$
101. $r, a, f, \theta.$	109. $a, b, \beta, \theta.$	117. $a-b, f, \beta, \theta.$
102. $r, f, g, \theta.$	110. $a, b, f, g.$	118. $f+a, b, c, \beta.$
103. $r, f, \alpha, \theta.$	111. $a, f, \alpha, \beta.$	119. $f-a, b, \beta, \gamma.$

D. Construction of tangent quadrilaterals.

When ρ is not given, apply Ex. 3, § 4. In some cases one side and the adjacent angles are easily constructed; the bisectors of these angles determine the centre of the inscribed circle.

In the case of the tangent trapezoid, the altitude of the trapezoid is equal to the diameter of the inscribed circle. The median passes through the centre of the inscribed circle; the same is true of the bisector of the angle formed by the two legs. If ρ and one of the legs are given, place between the bases a line equal in length to the given leg, and drop a perpendicular to this line from the centre of the inscribed circle; this perpendicular will meet the circle in the point of contact of the given leg.

To construct a tangent trapezoid, given :

120. $\rho, a, b.$	124. $a, b, \beta.$	128. $\rho, a+c, d.$
121. $\rho, b, \alpha.$	125. $a, \alpha, \beta.$	129. $\rho, a-b, c.$
122. $\rho, a, f.$	126. $a, b, c.$	130. $a-c, \alpha, \beta.$
123. $\rho, \alpha, \beta.$	127. $a, c, \alpha.$	131. $a+c, \alpha, \beta.$

To construct a tangent quadrilateral, given :

132. $\rho, \alpha, \beta, \gamma$.
133. ρ, a, b, γ.
134. ρ, a, f, α.
135. ρ, f, α, β.
136. ρ, a, b, α.
137. ρ, a, c, α.
138. ρ, c, α, β.
139. $\rho, a, c, \alpha+\beta$.
140. a, b, d, β.
141. a, b, c, f.
142. a, d, f, β.
143. a, b, α, β.
144. a, f, α, β.
145. a, f, β, γ.
146. $a+b, f, \alpha, \beta$.
147. $f+a, b, \beta, \gamma$.
148. $a+b, c, f, \beta$.
149. $f-a, b, d, \beta$.

§ 12. Miscellaneous Exercises.

1. If the altitude of an equilateral triangle is 6 inches, find the radii of the inscribed and the circumscribed circles.

2. If AC, BC are the legs of an isosceles triangle, and if the circles described upon AC, BC as diameters, meet in D, prove that CD bisects the angle ACB.

3. In every right triangle the line joining the vertex of the right angle to the centre of the square constructed upon the hypotenuse bisects the right angle.

4. In a triangle ABC the exterior angles at A, B, are bisected: Prove that the line which joins the intersection of the bisectors to the centre of the inscribed circle passes (if produced) through the vertex C.

5. How many circles can be described touching the sides of a triangle or the sides produced? How many lines of centres can be drawn? Prove that the circumference of the circumscribed circle passes through the middle point of every line of centres.

6. The feet of the altitudes and of the medians of a triangle lie in the same circumference.

7. To find the locus of the vertex of a triangle, given one side and the length of the corresponding median.

8. To construct a triangle, given the middle points of the sides.

9. To construct a triangle, given the three medians.

10. To construct a triangle, given h, m, t.

11. To construct a triangle, given the feet of the three altitudes.

12. To inscribe a rectangle in a circle, given $a+b$.

13. To inscribe a rectangle in a circle, given $a-b$.

14. To construct three equal circles about a given circle so that each shall touch the other two, and also the given circle.

15. To construct three equal circles within a given circle so that each shall touch two others, and also the given circle.

16. To construct four equal circles about a given circle so that each shall touch two others, and also the given circle.

17. To construct four equal circles within a given circle so that each shall touch two others, and also the given circle.

18. To construct three equal circles in an equilateral triangle so that each shall touch the other two, and also two sides of the triangle.

19. To construct four equal circles in a square so that each shall touch two others, and also two sides of the square.

20. In a given triangle to construct a semicircle having its diameter on one side, and touching the other two sides.

21. To inscribe a circle in a given sector.

22. To construct an equilateral triangle so that its vertices shall lie in three given parallel lines.

23. In a square $ABCD$ to construct an equilateral triangle AEF, so that E and F shall lie in the sides of the square.

24. To cut off the corners of a square by straight lines in such a way that a regular octagon shall be formed.

25. Through the vertices of a given equilateral triangle to draw lines which shall form another equilateral triangle having a given side.

26. In a given square to construct a square having a given side so that its vertices shall lie in the sides of the given square.

27. To inscribe a square in the part common to two equal intersecting circles.

28. Through a given point A in the plane of a given circle any secant ABC is drawn. At the middle point M of BC a perpendicular MP equal to MA is erected. Find the locus of the point P.

29. Given a circle and two parallel secants; to draw a tangent so that the part contained between the secants shall be bisected at the point of contact.

30. Given a circle and two lines, OA, OB, meeting at the centre O; to draw a tangent AB so that the part contained between the lines shall have a given length.

Given P, P_1, P_2; through P to draw a line so that, if perpendiculars P_1X, P_2Y are dropped to the line,

31. $PX+PY=a$. **32.** $PX-PY=b$.

CHAPTER III.

SIMILAR FIGURES.

§ 13. THEOREMS.

1. If three lines divide two parallels into proportional parts, these lines meet in one point.

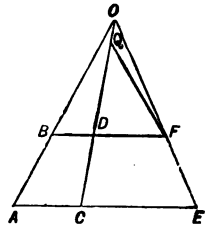


Fig. 45.

Let AB and CD meet in O (*Fig.* 45). In the similar $\triangle$ OAC, OBD, $AC : BD = CO : DO$.

Let CD and EF meet in Q. In the similar $\triangle$ QCE, QDF, $CE : DF = CQ : DQ$. But, by the hypothesis,

$$AC : BD = CE : DF.$$

Therefore, $$CO : DO = CQ : DQ;$$

a proportion from which it follows that O and Q must coincide. For, by the Theory of Proportions,

$$CO - DO : DO = CQ - DQ : DQ,$$

or, $$CD : DO = CD : DQ;$$

whence, $$DO = DQ.$$

2. Any two altitudes of a triangle are inversely proportional to the corresponding bases.

3. If the line joining the middle points of the bases of a trapezoid is produced, and the two legs are also produced, the three lines will meet in the same point.

4. If a line drawn from one vertex of a triangle divides the opposite side into parts proportional to the adjacent sides, the line bisects the angle at the vertex,

This theorem is the converse of No. 143. Either a direct or indirect proof may be given.

5. State and prove the converse of No. 144.

6. In a quadrilateral $ABCD$, in which the angles at B and D are right angles, perpendiculars PE, PF are dropped from any point P, in the diagonal AC, to the sides BC, AD, respectively. Prove that

$$\frac{PE}{AB}+\frac{PF}{CD}=1.$$

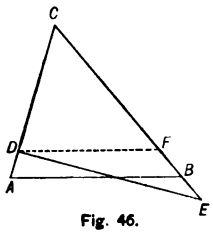


Fig. 46.

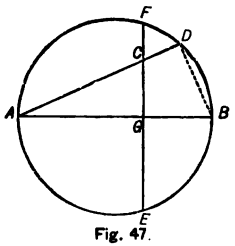


Fig. 47.

7. If in a triangle ABC any length AD is taken from one side AC, and an equal length BE is added to the side CB, the new base DE is divided by the base AB in the inverse ratio of the sides AC and BC.

Draw (*Fig.* 46) $DF \parallel$ to AB, and apply No. 139.

8. In a circle a line EF is drawn perpendicular to a diameter AB, and meeting it in G. Through A any chord AD is drawn, meeting EF in C. Prove that the product $AD \times AC$ is constant, whatever be the direction of AD.

Join BD (*Fig.* 47) and compare the $\triangle ACG$, ADB. Is the theorem also true when G lies outside the circle?

9. The squares of two chords, drawn through the same point in a circumference, have the same ratio as their projections upon the diameter drawn through that point. (No. 160.)

10. If two lines OA, OB, drawn through a point O, are divided in C, D, respectively, so that $OA \times OC = OB \times OD$, a circle can be described through the points A, B, C, D.

Show that (*Fig.* 48) the $\triangle DAO$, CBO are similar, and the $\angle DAO$ CBO equal. Therefore, if a segment be described upon CD capable of containing the $\angle DAO$, the arc of this segment will pass through B.

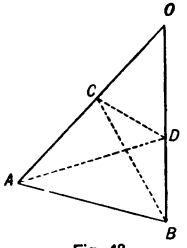


Fig. 48.

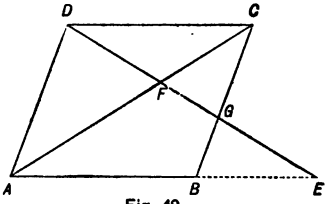


Fig. 49.

11. If in a parallelogram $ABCD$ a secant DE is drawn, meeting the diagonal AC in F, the side BC in G, and the side AB produced in E, then $DF^2 = FG \times FE$.

The $\triangle AFE$, DFC (*Fig.* 49) are similar; and also the $\triangle AFD$, CFG.

12. The sum of the squares of the segments formed by two perpendicular chords is equal to the square of the diameter of the circle.

(*Fig.* 50.) Apply No. 161; also show that $EC = AD$.

13. If three circles mutually intersect one another, the common chords pass through the same point.

Let M, N, R (*Fig.* 51) denote the circles, and let the chords CD, EF meet in O. Join AO, and suppose that AO produced does not pass through B, but through P in M and Q in N. Then we have,

In R, $OC \times OD = OE \times OF$.
In M, $OC \times OD = OA \times OP$.
In N, $OE \times OF = OA \times OQ$.
Whence, $OP = OQ$,

a relation which cannot be true unless P and Q coincide with B.

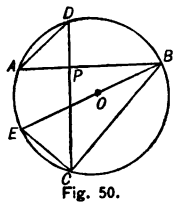


Fig. 50.

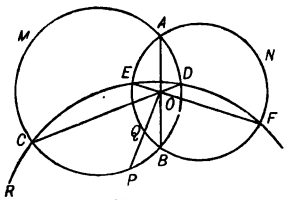


Fig. 51.

14. In every triangle the intersection of the three altitudes, the intersection of the three medians, and the intersection of the three perpendiculars erected at the middle points of the sides, lie in a straight line; and the distance between the first two points is double the distance between the last two.

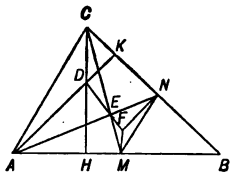


Fig. 52.

Let D, E, F (*Fig.* 52) be the three points in question. The line MN, joining the middle points of AB, BC, is $\parallel$ to AC. Hence, the $\triangle ADC$ and MNF are similar, and

$$\frac{NF}{AD} = \frac{MN}{AC} = \frac{1}{2}.$$

But $$\frac{NE}{AE} = \frac{1}{2}.$$ (§ 2, Ex. 43.)

Hence, the $\triangle ADE$ and NEF are similar, and $\angle AED = \angle NEF$; then DEF is a straight line. Also, from the $\triangle ADE$, NEF,

$$\frac{EF}{ED} = \frac{EN}{EA} = \frac{1}{2}.$$

15. Two circles cut in point P. Through P three lines are drawn, meeting one of the circles in A, B, C, the other in D, E, F, respectively. Prove that the triangles ABC, DEF are similar.

16. In every triangle the product of two sides is equal to the product of the diameter of the circumscribed circle and the altitude upon the third side.

If AC, BC are taken as the two sides, draw the diameter CE, and join EB.

17. In every inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.

Let $ABCD$ be the quadrilateral. In AC take a point E such that $\angle EDC = \angle ADB$. $\triangle ADB$ and CDE are similar; also, the $\triangle BCD$ and ADE. From these triangles obtain equations involving the sides, and then add them.

18. In every triangle the product of two sides is equal to the square of the bisector of the included angle plus the product of the segments into which it divides the third side.

If AC, BC are the two sides, and CD the bisector, produce CD to meet the circumscribed circle in E, and join BE. $\triangle ACD$ and ECB are similar; also, see No. 164.

19. In every triangle the sum of the squares of two sides is equal to twice the square of half the third side plus twice the square of the median drawn to the third side. (Nos. 162 and 163.)

20. In every triangle the difference of the squares of two sides is equal to twice the product of the third side and the projection of the median upon the third side.

21. In the diameter of a circle two points A, B are taken equally distant from the centre, and joined to a point P in the circumference. Prove that the sum $\overline{AP}^2 + \overline{BP}^2$ is constant for all positions of P. (Ex. 19.)

22. The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals. (Ex. 19.)

23. The sum of the squares of the sides of any quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals. (Ex. 19.)

24. The sum of the squares of the diagonals of a trapezoid is equal to the sum of the squares of the legs plus twice the product of the bases. (Ex. 23 and § 2, Ex. 56.)

25. Two triangles are similar if their sides are parallel each to each. Which are homologous sides?

26. Two triangles are similar if their sides are perpendicular each to each. Which are homologous sides?

27. If two similar triangles ABC, DEF have their homologous sides parallel, the lines AD, BE, CF which join their homologous vertices meet in the same point.

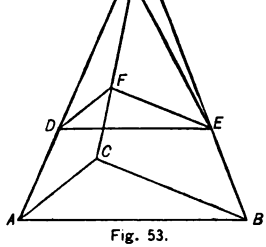


Fig. 53.

Let AD and CF (*Fig.* 53) meet in O, BE and CF in Q. From the similar triangles in the figure, prove that

$$CO : FO = CQ : FQ;$$

whence, $$CO - FO : FO = CQ - FQ : FQ.$$

Or, $$CF : FO = CF : FQ.$$

$$\therefore FO = FQ, \text{ or } Q \text{ coincides with } O.$$

28. Two polygons are similar if their homologous sides are parallel each to each.

29. If two similar polygons have their homologous sides parallel, the lines joining their homologous vertices meet in the same point.

Let two of the lines AA', BB' meet in O, and let AA' meet *any* third line, as DD', in Q. Draw diagonals, and prove, as in Ex. 24, that Q must coincide with O.

The ratio $AO : A'O$, $BO : B'O$, etc., are each equal to the ratio of any two homologous sides.

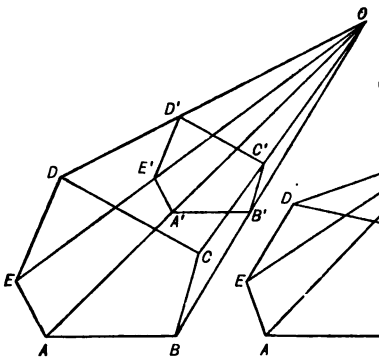


Fig. 54.

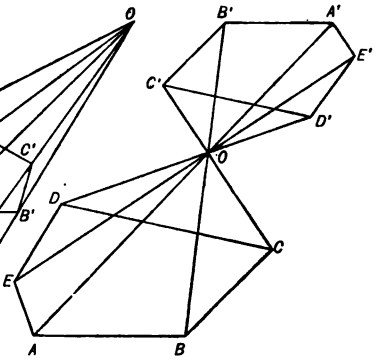


Fig. 55.

The point O is called the *centre of similitude* of the polygons. If the homologous sides are directed the same way (*Fig.* 54), the two polygons are said to be *similarly placed*, and O is called the *direct* centre of similitude. If the homologous sides are directed opposite ways (*Fig.* 55), the two polygons are said to be *inversely placed*, and O is called the *inverse* centre of similitude.

30. If a point O is joined to the vertices of a polygon $ABCDE$, and upon the lines OA, OB, etc., the lengths OA', OB', etc., are laid off, so that the ratios $OA' : OA$, $OB' : OB$, etc., are equal, the polygon $A'B'C'D'E'$ is similar to the polygon $ABCDE$.

31. If a point O is joined to the vertices of a polygon $ABCDE$, and through any point A' in OA a line parallel to AB is drawn, meeting OB in B', and through B' a line parallel to BC, meeting OC in C', etc., the polygon $A'B'C'D'E'$ is similar to the polygon $ABCDE$.

DEFINITIONS. Let AB be a given line, $m:n$ a given ratio, P a point in AB, such that $PA:PB = m:n$; then AB is said to be divided *internally* or *externally* in the ratio $m:n$, according as P is between A and B or in AB produced.

If the line joining the centres of two circles is divided externally and internally in the ratio of their radii, the points of division are called the *direct* and the *inverse* centres of similitude, respectively, of the two circles.

It follows from these definitions that the point of contact of two circles which touch *externally* is an *inverse* centre of similitude; and that the point of contact of two circles, one of which touches the other *internally*, is a *direct* centre of similitude.

32. The line joining the extremities of two parallel radii of two circles passes through the direct centre of similitude, if the radii have the same direction; and through the inverse centre if the radii have opposite directions.

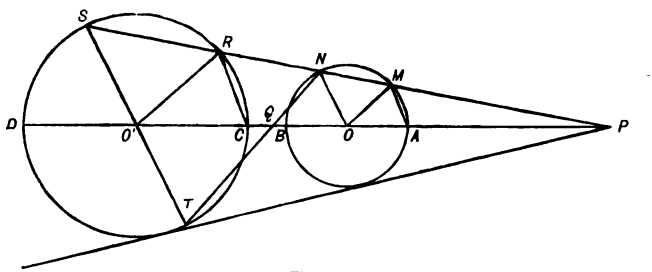


Fig. 56.

Let r, r' (*Fig.* 56) denote the radii of the circles, O, O' their centres; and let OM be $\parallel$ to $O'R$, $ON \parallel O'T$, RM and OO' meet in P. NT and OO' meet in Q. Then, $OP:O'P = r:r'$, and $OQ:O'Q = r:r'$. $\therefore$ P is the direct, and Q the inverse, centre of similitude.

33. The two radii of one circle, drawn to its points of intersection with any line passing through a centre of similitude, are parallel, respectively, to the two radii of the other circle, drawn to its intersections with the same line.

Proof indirect with the aid of Ex. 32.

34. All secants, drawn through the direct centre of similitude P of two circles, cut the circles in points whose distances from P, taken in order, form a proportion.

Let the line of centres (*Fig.* 56) cut the circles in the points A, B, C, D, and let any other secant through P cut the circles in the points M, N, R, S.

For the line of centres we have, by definition,

$$OP : O'P = r : r';$$

whence, $$OP - r : OP + r = O'P - r' : O'P + r'$$

or, $$PA : PB = PC : PD.$$

For any other secant we have, by similar triangles,

$$PM : PR = PN : PS = r : r'.$$

$$\therefore PM : PN = PR : PS.$$

35. Using the notation of the last exercise, prove that the product $PN \times PR$ is constant for all secants, and equal to the product $PB \times PC$.

Join MA and RC (*Fig.* 56), and show that the $\triangle$ PAM and PCR are similar; therefore,

$$PA : PM = PC : PR.$$

But $$PA : PM = PN : PB.$$

$$\therefore PN : PB = PC : PR,$$

and $$PN \times PR = PB \times PC.$$

36. The common exterior tangents to two circles pass through the direct centre of similitude, and the common interior tangents pass through the inverse centre of similitude.

What method of drawing the common tangents to two circles may be derived from this theorem?

37. Every straight line cutting the sides of a triangle (produced when necessary) determines upon the sides six segments, such that the product of three non-consecutive segments is equal to the product of the other three.

The line XYZ must cut either (i.) two sides of the triangle and the third side produced (*Fig.* 57), or (ii.) all three sides produced (*Fig.* 58). The proof for both cases is the same.

Draw $CD \parallel$ to AB. From the similar triangles,

$$\frac{AX}{CD} = \frac{AZ}{CZ}, \text{ and } \frac{BY}{CY} = \frac{BX}{CD}.$$

Therefore,
$$\frac{AX \times BY}{CY \times CD} = \frac{AZ \times BX}{CZ \times CD};$$

whence,
$$AX \times BY \times CZ = AZ \times BX \times CY.$$

This theorem was discovered by Menelaus of Alexandria, about 80 B.C.

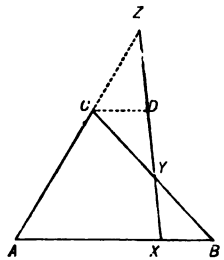


Fig. 57.

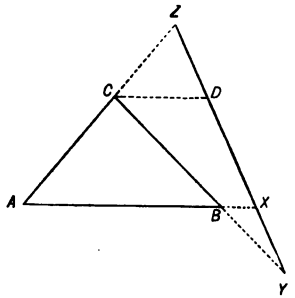


Fig. 58.

38. Prove the converse of the last theorem.

Let XY produced cut AC produced in a point P.

Then,
$$AX \times BY \times CP = AP \times BX \times CY.$$

But, by the hypothesis,

$$AX \times BY \times CZ = AZ \times BX \times CY;$$

whence,
$$CP : CZ = AP : AZ;$$

whence,
$$AP - CP : AZ - CZ = AP : AZ,$$

or,
$$AC : AC = AP : AZ.$$

$$\therefore AP = AZ;$$

that is, P coincides with Z.

39. Lines drawn through the vertices of a triangle, and passing through a common point, determine upon the sides six segments, such that the product of three non-consecutive segments is equal to the product of the three others.

The common point O may lie either within the triangle (*Fig.* 59), or without the triangle (*Fig.* 60). In both cases, apply Ex. 37 to the $\triangle ACD$ and line BOF, and to the $\triangle BCD$ and line AOE; then multiply the results.

This theorem was first made known by Ceva of Milan, in 1678.

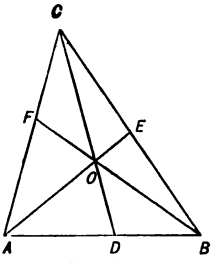


Fig. 59.

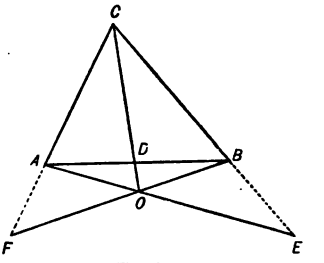


Fig. 60.

40. Conversely, if three lines drawn through the vertices of a triangle determine upon the sides six segments, such that the product of three non-consecutive segments is equal to the product of the three others, the lines pass through the same point.

The proof is similar to that of Ex. 38.

41. By means of Ex. 40, prove Ex. 29, § 2.

42. By means of Ex. 40, prove Ex. 32, § 2.

43. By means of Ex. 40, prove Ex. 43, § 2.

§ 14. Numerical Exercises.

1. Find the fourth proportional to the lines whose lengths are 25 feet, 32 feet, 48 feet.

2. Find the third proportional to the lines whose lengths are 36 feet and 24 feet.

3. Find the mean proportional between the lines whose lengths are 28 feet and 45 feet.

4. In a triangle ABC the side $AB = 305$ feet. If a line parallel to BC divides AC in the ratio $2 : 3$, what are the segments into which it divides AB?

Let x = one segment, then $305 - x$ = the other; and $x : 305 - x = 2 : 3$.

5. Upon two parallel lines six points are taken, A, B, C in one, and A', B', C' in the other, such that $AB = 2$ inches, $BC = 3$ inches, $A'B' = 1.24$ inches, $B'C' = 3.1$ inches. Will the lines AA', BB', CC' produced pass through the same point?

Find if the given numbers are in proportion.

6. The sides of a triangle are 9, 12, 15; find the segments of the sides made by the bisectors of the angles. (No. 143.)

7. If the sides of a triangle are denoted by a, b, c, find the segments of c made by the bisector of the opposite angle. What do the results become if $a = b$?

8. If the acute angles of a right triangle are 30° and 60°, what is the ratio of the segments into which the bisector of the angle of 60° divides the opposite side?

9. The sides of a triangle are 120, 80, 75. In a similar triangle the side homologous to 120 is equal to 90; find the other two sides.

10. Two lines start from a point A and cut two parallels. The first line cuts the parallels in B and C; the second line in D and E. If $BC = 4$ feet, $BD = 12$ feet, $CE = 18$ feet, $AE = 16$ feet, find AB, AD, DE.

11. At the ends of a line AB, perpendiculars AC, BD are erected, and in AB a point O is taken, such that the angles AOC, BOD are equal. If $AB = 25$ inches, $AC = 13$ inches, $BD = 7$ inches, find OA and OB.

Let $OA = x$, $OB = y$. The $\triangle AOC$, BOD (*Fig.* 61), are similar.

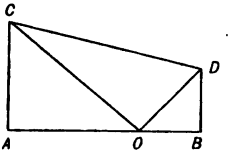


Fig. 61.

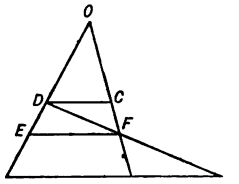


Fig. 62.

12. Given a trapezoid $ABCD$. The legs DA, CB are divided, starting from the points D, C, in the ratio $2:3$, and the points of division E, F are joined. Prove that EF is parallel to the bases, and compute the length of EF, if $AB = 3850$ feet, $CD = 1245$ feet.

Let AD, BC produced meet in O (*Fig.* 62); then

$$OA : OB = DA : CB.$$

Also, $$DE : EA = CF : FB;$$

whence, $$DA : CB = EA : FB.$$

$$\therefore OA : OB = EA : FB.$$

$$\therefore EF \text{ is } \parallel \text{ to } CD.$$

Draw DFG; then $BG : DC = 3 : 2$, and $EF : AG = 2 : 5$.

13. If in the trapezoid $ABCD$ (*Fig.* 62) the bases AB, CD are denoted by a, b, respectively, and the altitude by h, find the altitude of the triangle AOB formed by producing the legs.

14. A tree casts a shadow 60 feet long at the same time that a vertical rod 3 feet high casts a shadow 2 feet long; find the height of the tree.

15. Show how to find, by means of similar triangles, the distance AX (*Fig.* 63). If $AP = 200$ feet, $OP = 20$ feet, $OQ = 32$ feet, find AX.

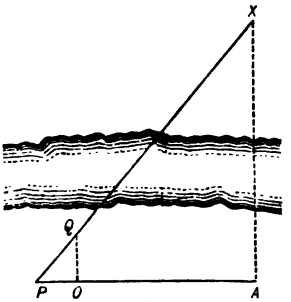


Fig. 63.

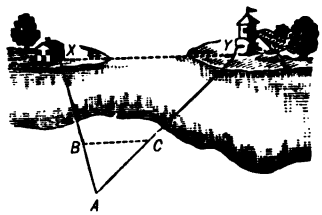


Fig. 64.

16. Show how to find the distance between two inaccessible objects X, Y. (*Fig.* 64.) If $AX = 4$ miles, $AY = 5$ miles, $AB = 200$ feet, $AC = 250$ feet, $BC = 225$ feet, find XY.

17. The perimeters of two similar polygons are 280 feet and 160 feet. If a side of the first polygon is 15 feet, find the homologous side of the second.

18. Required the length of a ladder which will reach a window 24 feet high, if the lower end of the ladder is 10 feet from the side of the building.

19. How far apart are the opposite corners of a floor 12 feet by 16 feet?

20. If the side of an equilateral triangle is a, find the altitude.

21. Find the legs of a right triangle if their projections upon the hypotenuse are 2.88 feet and 5.12 feet.

$x^2 : y^2 = 2.88 : 5.12$, and $x^2 + y^2 = 64$.

22. The legs of a right triangle are 10 feet and 24 feet. Find their projections upon the hypotenuse, and the altitude upon the hypotenuse.

23. Given the legs a, b of a right triangle, find their projections x, y on the hypotenuse, and the altitude h upon the hypotenuse.

24. The hypotenuse of a right triangle is 1, and the sum of the legs is 1.4; find the legs.

25. Find the three sides of a right triangle if these sides are three consecutive integral numbers.

Denote the sides by $x-1$, x, $x+1$.

26. Compute the legs of a right triangle if their ratio is 3 : 4, and the hypotenuse is 40.

27. Find the sides of a right triangle if their sum is 132 and the sum of their squares is 6050.

28. If in a right triangle the hypotenuse is 25 feet, one leg is 15 feet; find the altitude upon the hypotenuse.

29. The legs of a right triangle are 3.128 and 4.275; compute to 0.001 of a unit the segments of the hypotenuse made by the bisector of the right angle.

30. The radius of a circle is 5 inches. Find the distance from the centre to a chord 8 inches long.

31. The radius of a circle is r. What is the length of a chord the distance of which from the centre is $\frac{1}{2}r$? What angle does it subtend at the centre?

32. If the radii of two concentric circles are 10 inches and 8 inches, find the length of a chord in the larger circle which touches the smaller circle.

33. Two circles, the radii of which are r, r', intersect at right angles; find the distance between their centres.

The tangents at the point of intersection are perpendicular to each other, and form with the line of centres a right triangle.

34. The radii of two circles are 8 inches and 3 inches, and the distance between their centres is 15 inches. Find the length of their common exterior tangent.

35. The radius of a circle is 6 inches. Through a point 10 inches from the centre, tangents are drawn. Find the lengths of these tangents and of the chord of contact.

36. The distance between two parallel lines is a, and the distance between two points A, B in one of the lines is $2b$. Find the radius of a circle which passes through A and B, and touches the other line. What is its value if $a = b$?

37. The sides of a triangle are 8, 9, 13; is the greatest angle acute, right, or obtuse?

38. If in an isosceles triangle a denote one of the equal sides, and b half the base, find the radius of the circumscribed circle.

Draw the altitude, and drop a perpendicular from the centre of the circle to one of the equal sides; two similar triangles are formed.

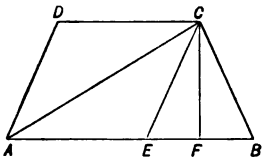


Fig. 65.

39. In an isosceles trapezoid let a = the greater base, b = the other base, c = one of the legs; find the lengths of the diagonals.

The two diagonals are equal. (Ex. 51, § 2.) Draw (*Fig.* 65) $CE \parallel$ to DA, $CF \perp$ to AB; in the isos. $\triangle CBE$, $FB = \frac{1}{2}EB = \frac{1}{2}(a - b)$. Then apply No. 162 to the $\triangle ACB$.

40. Compute the sides of a rectangle, given a diagonal d and the perimeter $2p$. When is the problem possible? when impossible?

Let x and y denote the two sides; then $x+y=p$, $x^2+y^2=d^2$; whence,

$$x=\frac{p+\sqrt{2d^2-p^2}}{2},$$

$$y=\frac{p-\sqrt{2d^2-p^2}}{2}.$$

If $p^2<2d^2$, or $p<d\sqrt{2}$, the problem is possible; if $p^2>2d^2$, or $p>d\sqrt{2}$, the roots are imaginary, and the problem is impossible. If $p=d\sqrt{2}$, then $x=y=\frac{1}{2}p$, and the rectangle is a square.

41. Two chords AB, CD intersect in M; if $AM=5$ inches, $BM=6$ inches, $CD=11.5$ inches, find CM and MD.

$x+y=11.5$, $xy=30$.

42. Two chords AB, CD intersect in M; if $AM=4$ inches, $BM=5$ inches, and the difference between CM and $MD=8$ inches, find CD.

43. The diameter of a circle is equal to 30 feet, and is divided into three equal parts; find the lengths of the perpendiculars drawn from the points of division and terminated by the circumference.

44. What must be the distance of a point from the centre of a circle (radius $=r$) in order that a tangent drawn from a point to the circle may be equal to three times the radius?

45. Through a point P, exterior to a circle, a tangent PA and a secant PBC are drawn; if $PB=5$ inches, $BC=4$ inches, find PA.

46. Find in a line AB touching a circle (radius $=r$) in A, a point C, such that the exterior part CD of the line which joins C to the centre shall be equal to $\frac{1}{2}AC$.

Let $x = AC$ (*Fig.* 66); then $\frac{x}{2} = CD$, and $x^2 = \frac{x}{2}\left(2r + \frac{x}{2}\right)$; whence, $x = 0$ or $\frac{4}{3}r$.

47. The radius of a circle equals 2 inches. Through a point A, 4 inches from the centre, a secant ABC is drawn. If $BC = 1$ inch, find AB.

Let $x = AB$ (*Fig.* 67); then $x(x+1) = 12$, whence $x = 3$ or -4. The negative root must be rejected, or else considered as the solution of the question obtained by changing x to $-x$ in the equation $x(x+1) = 12$. The equation then becomes $x^2 - x = 12$, and belongs to the question, find the length of the secant AC if $BC = 1$ in.

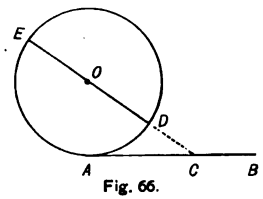


Fig. 66.

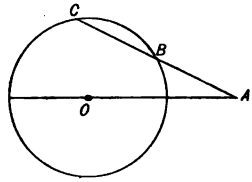


Fig. 67.

48. In a triangle given the sides, $a = 1551$ feet, $b = 2068$ feet, $c = 2585$ feet; find the median to the side c.

If x equal the required line, then (Ex. 19, § 13) $x^2 + b^2 = 2x^2 + \frac{1}{2}c^2$. In this particular case the labor of computing x may be avoided by observing that the given numbers are equimultiples of 3, 4, 5, respectively; therefore the triangle is a right triangle (No. 161), and $x = \frac{1}{2}c$. (Ex. 45, § 2.)

49. In a triangle given the sides a, b, c; find the lengths of the three medians. (Ex. 19, § 13.)

If x, y, z denote the lengths, then

$$x = \tfrac{1}{2}\sqrt{2a^2 + 2b^2 - c^2};$$

$$y = \tfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2};$$

$$z = \tfrac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}.$$

By adding the squares of these values, we obtain

$$x^2 + y^2 + z^2 = \tfrac{3}{4}(a^2 + b^2 + c^2);$$

or, the sum of the squares of the medians is equal to three-fourths of the sum of the squares of the sides.

50. In a triangle given the sides a, b, c; find the lengths of the three bisectors.

Let x, y, z (*Fig.* 68) denote the lengths of the bisectors of the angles opposite c, a, b, respectively, and let x divide the side c into the segments u, v. Then $ab = x^2 + uv$ (Ex. 18, § 13), $u : v = a : b$ (No. 143), and $u + v = c$.

By eliminating u and v, we have

$$x^2 = \frac{ab[(a+b)^2 - c^2]}{(a+b)^2} = \frac{ab(a+b+c)(a+b-c)}{(a+b)^2}$$

Let $a + b + c = 2p$; then $a + b - c = 2(p - c)$, and

$$x = \frac{2\sqrt{abp(p-c)}}{a+b}$$

Similarly,
$$y = \frac{2\sqrt{bcp(p-a)}}{b+c};$$

$$z = \frac{2\sqrt{acp(p-b)}}{a+c}.$$

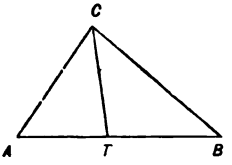


Fig. 68.

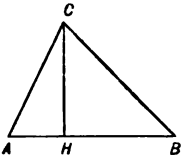


Fig. 69.

51. Given in a triangle ABC the sides a, b, c; find the segments of AB made by the altitude upon AB.

Let $x = AH$ (*Fig.* 69); then $c - x = BH$.

Then $$\overline{CH}^2 = a^2 - (c-x)^2,$$

also $$\overline{CH}^2 = b^2 - x^2,$$

$$\therefore a^2 - (c-x)^2 = b^2 - x^2,$$

whence $$x = \frac{b^2 + c^2 - a^2}{2c},$$

and $$c - x = c - \frac{b^2 + c^2 - a^2}{2c} = \frac{a^2 + c^2 - b^2}{2c}$$

52. In a triangle given the sides a, b, c; find the lengths of the three altitudes.

Let x, y, z denote the lengths of the altitudes upon c, a b, respectively. By the last Exercise,

$$x^2 = b^2 - \overline{AH}^2 = b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2 = \frac{4b^2c^2-(b^2+c^2-a^2)^2}{4c^2}.$$

$$\therefore x^2 = \frac{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}{4c^2}$$

$$= \frac{[(b+c)^2-a^2][a^2-(b-c)^2]}{4c^2}$$

$$= \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4c^2}$$

Let $a+b+c=2p$; then $b+c-a=2(p-a)$, $(a-b+c)=2(p-b)$, $(a+b-c)=2(p-c)$. Hence,

$$x^2 = \frac{16p(p-a)(p-b)(p-c)}{4c^2};$$

$$x = \frac{2\sqrt{p(p-a)(p-b)(p-c)}}{c}.$$

Similarly,

$$y = \frac{2\sqrt{p(p-a)(p-b)(p-c)}}{a};$$

$$z = \frac{2\sqrt{p(p-a)(p-b)(p-c)}}{b}.$$

§ 15. Loci.

1. *Notation.* The same notation is used as in Chapter II. Also, PH denotes the perpendicular from P to L.

2. To find the locus of points which divide lines drawn from P to L in the ratio $m : n$.

The locus is the line parallel to L which divides PH in the given ratio. Special case: $PH = 4$ inches, $m = 3$, $n = 1$.

3. To find the locus of the ends of lines which are drawn from P, and are divided by L in the ratio $m : n$.

Special case: $PH = 3$ inches, $m = 3$, $n = 2$.

4. To find the locus of the ends of lines which are drawn from L to P, and then produced so that they are divided by P in the ratio $m : n$.

Special case: $PH = 2$ inches, $m = 4$, $n = 3$.

5. To find the locus of points the distances of which from two given parallels are as $m : n$.

Special case: $m = 2$, $n = 1$.

6. To find the locus of points the distances of which from two intersecting lines L and L' are as $m : n$.

The locus consists of two straight lines. Draw parallels to L and L', such that their distances from L and L' respectively shall be as $m : n$; these parallels will intersect in points belonging to the required locus. Special case: $\angle LL' = 60°$, $m = 2$, $n = 1$.

7. Between the sides of a given angle a series of parallels are drawn; to find the locus of points which divide these parallels in the ratio $m : n$.

Special case: $m = 4$, $n = 1$.

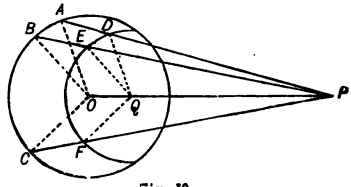


Fig. 70.

8. Through P secants are drawn to a circle K; to find the locus of points which divide the entire secants in the ratio $m : n$.

Through points of the locus D, E, F (*Fig.* 70), draw lines parallel respectively to the radii OA, OB, OC; these lines divide PO in the given ratio, and therefore meet in a point Q in PO. From the similar triangles, $QD : QE : QF = OA : OB : OC$. But $OA = OB = OC$. Therefore $QD = QE = QF$. Therefore the locus is an arc with Q as centre and QD as radius.

Special cases: (i.) P in K, and $m = n$; (ii.) P within K, and $m = 2n$.

9. To find the locus of a point such that the sum of the squares of its distances from two given points A, B is constant.

The locus is a circle having for centre the middle point of the line AB. If P (*Fig.* 71) denote any point of the locus, k^2 the constant quantity, m the median PM of the $\triangle ABP$, then $AP^2 + BP^2 = k^2$; also (Ex. 19, § 13), $\overline{AP}^2 + \overline{BP}^2 = 2m^2 + 2\overline{AM}^2$;

whence, $$m = \sqrt{\frac{k^2 - 2\overline{AM}^2}{2}},$$ a constant quantity.

If $k^2 < 2\overline{AM}^2$, or $k < AM\sqrt{2}$, there is no locus; if $k^2 - 2\overline{AM}^2 = 0$, or $k = AM\sqrt{2}$, the locus is reduced to a point.

Special case: $AB = 4$ inches, $k^2 = 100$.

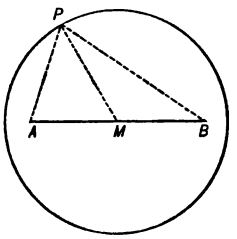


Fig. 71.

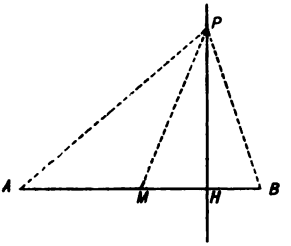


Fig. 72.

10. To find the locus of a point such that the difference of the squares of its distances from two given points A, B is constant.

The locus consists of two straight lines perpendicular to the line AB. If P (*Fig.* 72) is a point of the locus, k^2 the constant quantity, H the foot of the perpendicular from P to AB, then $\overline{AP}^2 - \overline{BP}^2 = k^2$; also (Ex. 20, § 13), $AP^2 - BP^2 = 2AB \times MH$;

whence, $$MH = \frac{k^2}{2AB},$$ a constant quantity.

Therefore, the perpendicular to AB erected at H is one part of the locus, and the distance MH is a third proportional to the lengths $2AB$ and k. The other part of the locus is the corresponding perpendicular erected at the distance MH on the other side of M.

Special case: $AB = 6$ inches, $k^2 = 64$.

14. To find the locus of points such that the tangents drawn from each point to two given circles shall be equal.

The locus is a line perpendicular to the line of centres, and is called the *radical axis* of the two circles. Let P (*Fig.* 77) be a point in the locus. Then, $PA = PB$; whence, from the rt. $\triangle$ POA, PQB,

$$\overline{PO}^2 - \overline{AO}^2 = \overline{PQ}^2 - \overline{QB}^2;$$

or, $$\overline{PO}^2 - \overline{PQ}^2 = \overline{AO}^2 - \overline{QB}^2, \text{ a constant quantity.}$$

This reduces the problem to Ex. 10, and it follows that the locus is a line perpendicular to OQ, and cutting OQ in a point H, found as in Ex. 10. If the circles are exterior to each other, the shortest way to construct the locus is to draw a common tangent TS, bisect it in C, and then draw $CH \perp$ to OQ.

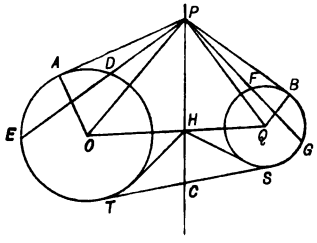


Fig. 77.

If the two circles touch each other, the radical axis is the common interior tangent; if they cut each other, the radical axis is the common chord. For, if the radical axis meets one of the circles, it must meet the other in the same point, in order that the two tangents drawn from this point may have the same value, namely, zero.

If through P a secant PDE is drawn to one of the circles, and a secant PFG to the other, then $PD \times PE = PA^2$, and $PF \times PG = PB^2$. $\therefore PD \times PE = PF \times PG$; that is, the radical axis is also the locus of points of equal power with respect to the two circles.

It is evident that a circle with P as centre, and PA as radius, will cut both the given circles at right angles: so that the radical axis is also the locus of the centres of all circles which cut two given circles at right angles.

Special cases: (i.) $OQ = 6$ inches, $r = 3$ inches, $r' = 2$ inches; (ii.) $OQ = 6$ inches, $r = r' = 2$ inches.

15. To find the locus of points such that the tangent drawn from each point to a given circle shall be equal to the distance of the point from a given point P.

This is a special case of Ex. 14; for the reasoning is independent of the size of the two circles, and holds true if one of the circles is reduced to a point. Therefore, the locus is a line perpendicular to the line which joins P to the centre of the given circle. If P is without the circle, the locus bisects the tangent drawn from P; if P is within the circle, the locus bisects the two tangents whose chord of contact is bisected by P. (Proof.)

Special cases: (i.) $OP = 5$ inches, $r = 4$ inches; (ii.) $OP = 4$ inches, $r = 4$ inches.

16. To find the locus of points from which two given circles will be seen under equal angles.

Show that the distances from any point in the locus to the centres of the two circles are as the radii of the circles; this reduces the problem to Ex. 12.

17. To find the locus of the points from which a given straight line is seen under a given angle.

18. To find the locus of the vertex of a triangle, having given the base and the ratio of the other two sides.

19. To find the locus of the points in a plane equally illuminated by two lights A, B placed in the plane; given that the intensities of the two lights at unit distance are as $m : n$, and that the intensity varies inversely as the square of the distance.

Special case: $AB = 6$ inches, $m = 2$, $n = 1$.

20. Through a point A a line is drawn meeting a given circle in C. In this line a point P is taken such that $AP \times AC = k^2$. Find the locus of the point P.

21. A given point O is joined to point M in a given line. Upon the line OA a length OP is taken such that $OP \times OA = k^2$. Find the locus of the point P.

§ 16. Problems.

1. To construct the mean proportional between two given lines by three different methods, and to show that the methods verify one another. (Nos. 159, 160, 166.)

2. Explain how a line 26 inches long may be divided into three parts proportional to the numbers 2, $\frac{3}{4}$, $\frac{1}{2}$. Compute the three parts.

To divide a line AB in a point C so that,

3. $AC \times CB = k^2$. **4.** $AB \times AC = k^2$.

5. To construct two lines, given their sum and their ratio.

6. To construct two lines, given their difference and their ratio.

7. To cut from two lines equal lengths such that the remaining parts shall have the ratio $m : n$.

8. To produce a line AB to a point C such that $AB : AC = 3 : 5$.

9. To construct a line, given the greater segment of the line divided in extreme and mean ratio.

10. To divide a line AB *harmonically* in a given ratio $m : n$; that is, to find a point C in AB and a point D in AB produced such that $AC : BC = AD : BD = m : n$.

See solution of Ex. 12, § 15, or apply Nos. 143 and 144. A and B are called *conjugate* points; likewise, C and D. Examine the case when $m = n$.

11. Given in the harmonic division of a line three points A, B, C; to find the fourth point D.

12. To construct a triangle similar to a given triangle by applying Ex. 29, § 13.

Special case: ratio of similitude $= \frac{1}{2}$.

13. To draw the common tangents to two circles by applying the properties of their centres of similitude.

14. To draw a line from P to L which shall be to the perpendicular dropped from P to L as $m : n$.

Given an angle BAC and a point P; through P to draw a line meeting the sides of the angle in X and Y, so that,

15. $AX : AY = m : n$. **16.** $PX : PY = m : n$.

In Ex. 15 and 16 take P (i.) within the angle, and also (ii.) without the angle.

17. Three given lines meet in a point P; through P to draw a line such that the two segments made by the given lines may have a given ratio.

Through any point in the middle one of the given lines, draw a line so that its segments may have the given ratio; then, through P draw a parallel.

Given P and L; to find a point X such that PX shall be cut by L in a given ratio, and

18. X shall be at the distance a from P_1.

19. X shall be at the distance a from L_1.

20. X shall be equidistant from P_1 and P_2.

21. X shall be equidistant from L_1 and L_2.

22. In one side of a given triangle to find a point, the distances of which from the other sides shall have a given ratio.

23. Within a given triangle to find a point the distances of which from the three sides shall be as the numbers $m : n : p$.

24. Between two parallel tangents to a circle, to draw a third tangent so that it shall be divided by the point of contact in a given ratio.

25. Given L, L_1, K; to construct a circle having a given radius, touching K, and having the distances from its centre to L and L_1 in a given ratio.

26. Given P, $L \parallel$ to L_1, and L_2 cutting L and L_1; through P to draw a line from which the given lines shall cut two segments having a given ratio.

27. In K given P and P_1; to place in K a chord of given length so that the distances from P and P_1 to the chord shall be as 3 : 1.

To find a point X such that the distances from two given points shall have a given ratio, and

28. X shall be at the distance a from L.

29. X shall be equidistant from L and L_1.

30. The tangent from X to K shall have a given length.

31. The distances from X to two other given points shall also have a given ratio.

32. In a given triangle, to find a point the distances of which from the three vertices shall be as the numbers $m : n : p$.

33. To find a point from which the lengths AB, BC, CD taken in a straight line shall be seen under equal angles. (No. 143, and § 15, Ex. 12.)

34. To find a point from which three given circles will be seen under equal angles.

The distances of the required point from the centres of the circles are proportional to the respective radii of the circles. (Ex. 16, § 15.)

35. In a diameter of a circular billiard table are placed two balls; to find by construction in what direction one of the balls must be struck centrally in order that it may hit the other after first striking the side of the table.

36. To find the point which is equally illuminated by three lights situated in the same plane. (Ex. 19, § 15.)

37. To inscribe a square in a semicircle.

Suppose the problem solved, and $ABCD$ (*Fig.* 78) the square required. First prove that $OA = OB = \frac{AD}{2}$; then draw $EH \perp$ to EF, and meeting OD produced in H; we have $\frac{OE}{EH} = \frac{OA}{AD} = \frac{1}{2}$, whence $EH = 2\,OE$.

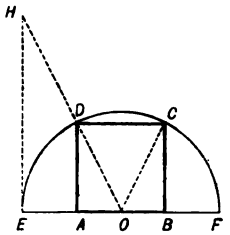


Fig. 78.

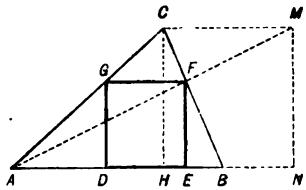


Fig. 79.

38. To inscribe a square in a given triangle.

Suppose the problem solved, and $DEFG$ (*Fig.* 79) the inscribed square. Draw $CM \parallel$ to AB, and let AF produced meet CM in M; then $GF : CM = AF : AM$. Draw $MN \perp$ to AB, and $CH \perp$ to AB; then $FE : MN = AF : AM$. Now $GF = FE$; $\therefore CM = MN = CH$. Therefore, construct a square upon CH as one side, and the line AM will determine the point F.

39. To inscribe in a triangle a rectangle similar to a given rectangle.

The solution is like that of Ex. 38; upon CH (*Fig.* 79) construct a rectangle similar to the given rectangle.

40. To inscribe in a semicircle a rectangle similar to a given rectangle.

41. To inscribe in a circle a triangle similar to a given triangle.

42. To circumscribe about a circle a triangle similar to a given triangle.

In the triangle ABC to draw a parallel to AB, meeting AC in X and BC in Y, so that,

43. $AB : XY = m : n$. **44.** $AC : XY = XY : CX$.

From the similar $\triangle ABC$, XYC we have $AB : AC = XY : CX$; $\therefore AB : AC = AC : XY$.

45. $AB : XY = XY : CX$.

46. $BC : XY = XY : CX$.

47. $AB : XY = AX : BY$.

48. $AX : XY = XY : BY$.

Through C draw a line $\parallel$ to AB; produce AY and BX to meet it.

49. Given P within K; through P to draw a chord XY so that $PX : PY = m : n$. If $r = 4$ inches, $PO = 2$ inches, between what limits must the ratio $m : n$ lie in order that the problem may be possible?

Draw $AY \parallel$ to OX and meeting OP in A; then $OP : PA = m : n$, also $r : AY = m : n$.

50. Given P in the arc AB of K; to draw a chord PX which shall be divided by the chord AB in the ratio $m : n$.

Draw $XC \parallel$ to AB and meeting PO.

51. Given K, a chord AB, P in AB; through P to draw a chord XY so that $AX : BY = m : n$.

52. Given K and a chord AB; to find a point X in K so that $AX : BX = m : n$.

53. In K to draw a chord which shall be divided by two given radii into three equal parts.

54. Given P in K; from P to draw two chords PX, PY so that $PX : PY = m : n$, and XY shall be a diameter.

Through P draw a line $\perp$ to PO; two similar $\triangle$ are formed.

55. In a given arc AB to find a point X such that $AX \times BX = k^2$.

Given P without K; through P to draw a secant PXY, so that,

56. $PX : XY = XY : PY$.

Draw a tangent from P to the circle.

57. $XY = 2PX$.

58. $PX \times XY = k^2$. **60.** $PY : PX = PX : XY$.

Draw a tangent PA, also $BX \parallel$ to AY; then PA is divided by B in extreme and mean ratio; also from the similar $\triangle$s PAX, PBX we have $PX^2 = PA \times PB$.

61. Given K and two radii OA, OB; to draw a tangent CXD, touching K at X and meeting OA, OB produced in C, D, so that $CX : XD = m : n$.

Draw $AE \parallel$ to CD, meeting OX in E and OD in F, and draw $EG \parallel$ to OD meeting OA in G; then $AG : GO = m : n$.

62. Given K and K_1 intersecting in P; through P to draw a line so that the chords intercepted by the circles shall be as $m : n$.

A line through P perpendicular to the required line will divide the line of centres in the ratio $m : n$.

63. In L to find a point X from which the tangents drawn to K and K_1 shall be equal.

64. To find a point X from which the tangents drawn to three given circles shall be equal.

65. To construct a circle having a given radius, and cutting two given circles at right angles.

66. To construct a circle which shall pass through P and cut K at right angles in P_1.

67. To construct a circle which shall pass through P and cut K and K_1 at right angles.

68. To construct a circle which shall cut three given circles at right angles.

§ 17. The Method of Similar Figures.

1. This method of constructing figures is based upon the fact that the *shape* of a figure is determined by certain angles or the ratios of certain lines. First construct a figure *similar* to the figure required, and then construct the required figure by means of the general theorem that in similar figures *homologous lines are proportional.*

2. The shape of a triangle is determined by:

(i.) two angles;

(ii.) the ratio of two sides and the included angle;

(iii.) the ratios of the three sides;

(iv.) the ratios of the three altitudes;

(v.) the ratio of an altitude to the corresponding base, and the angle at the vertex.

3. This method is also applicable to cases in which the shape, not of the required figure, but of an auxiliary triangle, is determined by the given parts; as, for example, by:

(i.) one angle;

(ii.) the ratio of an altitude to one of the adjacent sides;

(iii.) the ratio of an altitude to the corresponding median;

(iv.) the ratio of the radius of the circumscribed circle to one side.

4. The notation used in the following exercises is the same as in Chapter II.

5. To construct a triangle, given α, β, $h + m$.

Analysis. Any $\triangle DEC$ (*Fig.* 80), in which $\angle CDE = \alpha$, $\angle CED = \beta$, will be similar to the triangle required. If in this triangle we draw the altitude CF and the median CG, then will each side of the required triangle be a fourth proportional to three known lengths, $CF + CG$, $h + m$, and the homologous side of the $\triangle DEC$. The problem is then reduced to the construction of a triangle from two angles and one side.

Construction. Upon any line DE construct a $\triangle DEC$, making $\angle CDE = \alpha$, $\angle CED = \beta$. Draw $CF \perp$ to DE, bisect DE in G, and join CG. Produce CF to L, making $FL = CG$, and upon CL take $CK = h + m$. Join LD, and through K draw $KA \parallel$ to LD, cutting CD (or CD produced) in A. Then draw $AB \parallel$ to DE, and meeting CE (or CE produced) in B. $\triangle ABC$ is the triangle required.

Proof. $\angle CAB = \angle CDE = \alpha$, $\angle CBA = \angle CED = \beta$.

Let AB cut CF in H, CG in M. CH is $\perp$ to AB, M is middle point of AB.

$$CH : CF = CM : CG = CA : CD,$$

$$CH + CM : CF + CG = CA : CD,$$

$$CA : CD = CK : CL.$$

$$\therefore CH + CM : CF + CG = CK : CL.$$

But $CF + CG = CL$, and $CK = h + m$.

$$\therefore CH + CM : CL = h + m : CL.$$

Whence, $CH + CM = h + m$.

Discussion. There is always one, and only one, solution.

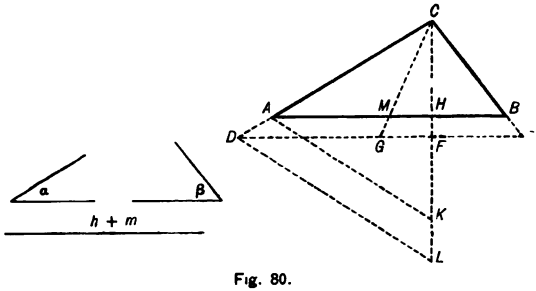


Fig. 80.

6. To construct a triangle, given $h : a$, $c + h$, a.

Analysis. Let ABC (*Fig.* 80) be the triangle required, CH its altitude h. The given ratio $h : a$ determines the shape of the rt. $\triangle CBH$. Therefore, if we construct two lines m and n, so that $m : n = h : a$, and then construct a rt. $\triangle CEF$, making $CF = m$, $CE = n$, the rt. $\triangle CEF$ will be similar to the rt. $\triangle CBH$; and if we draw through C a line CD, making $\angle CDE = \alpha$, we shall have a $\triangle CDE$ similar to the triangle required. The remainder of the solution is precisely similar to that of the last exercise.

7. To construct a triangle, given the three altitudes h, h_a, h_b.

Analysis. By Ex. 2, § 13, $a : c = h : h_a$, and $b : c = h : h_b$; hence,

$$a = \frac{ch}{h_a}, \quad b = \frac{ch}{h_b}. \quad \therefore a : b : c = \frac{h}{h_a} : \frac{h}{h_b} : 1;$$

that is, the sides of the triangle required are proportional to the numbers

$$\frac{h}{h_a}, \ \frac{h}{h_b}, \text{ and } 1;$$

and any $\triangle DEC$ (*Fig.* 81), whose sides are proportional to these numbers, will be similar to the triangle required. By drawing $CF \perp$ to DE, taking $CH = h$, and drawing through H, $AB \parallel$ to DE, we obtain the required $\triangle ABC$.

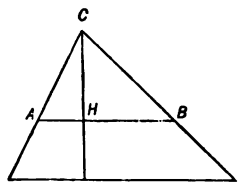


Fig. 81.

To construct a triangle, given :

8. Rt. $\triangle$, a, m.
9. Rt. $\triangle$, a, $c+h$.
10. Rt. $\triangle$, a, $c-h$.
11. Rt. $\triangle$, a, ρ.
12. Isos. $\triangle$, γ, $c+h$.
13. Isos. $\triangle$, γ, $c+r$.
14. Isos. $\triangle$, a, $a-\rho$.
15. Isos. $\triangle$, a, $r+\rho$.
16. a, β, $c-h$.
17. a, β, $r+\rho$.
18. a, β, $r-\rho$.
19. a, β, $r+h$.
20. a, β, $c+t$.
21. $a : b$, $c+h$, γ.
22. $a : b$, $c-h$, γ.
23. $a : b$, $r+\rho$, γ.

24. $a : b,\ r + h,\ \gamma.$
25. $a : b,\ h + \rho,\ \gamma.$
26. $a : b,\ c + t,\ \gamma.$
27. $a : b,\ c + m,\ \gamma.$
28. $a : b,\ m + h,\ \gamma.$
29. $a : b,\ b : c,\ h.$
30. $a : b,\ b : c,\ p.$
31. $h_a : h_b,\ r,\ \gamma.$
32. $h_a : h_b,\ \rho,\ \gamma.$
33. $h_a : h_b,\ h,\ \gamma.$
34. $h_a : h_b,\ m,\ \gamma.$
35. $h : h_a,\ h : h_b,\ r.$
36. $h : h_a,\ h : h_b,\ \rho.$
37. $h : h_a,\ h : h_b,\ m.$
38. $h : h_a,\ h : h_b,\ t.$
39. $h : a,\ h : b,\ c.$
40. $h : a,\ h : b,\ m.$
41. $h : a,\ h : b,\ t.$
42. $h : a,\ b,\ \alpha.$
43. $h : a,\ c,\ \alpha.$
44. $h : a,\ b + c,\ \alpha.$
45. $h : a,\ a + c,\ b.$
46. $h : a,\ c,\ r.$
47. $h : c,\ a + b,\ \gamma.$
48. $h : c,\ a - b,\ \gamma.$
49. $h : c,\ h + \rho,\ \gamma.$
50. $h : m,\ a + b,\ \alpha.$
51. $h : m,\ a + c,\ \alpha.$
52. $h : m,\ h + c,\ \alpha.$
53. $h : m,\ m + c,\ \alpha.$
54. $h : m,\ a,\ \beta.$
55. $h : m,\ a,\ c.$
56. $h : m,\ c,\ \beta.$
57. $h : a,\ a : c,\ h_a.$
58. $h : a,\ a : c,\ t.$
59. $h : a,\ a : c,\ r + \rho.$
60. $r : a,\ c,\ \beta.$
61. $r : a,\ \rho,\ \beta.$
62. $r : a,\ b + c,\ \beta.$
63. $r : a,\ b : c,\ h_a.$
64. $r : a,\ b : c,\ r + \rho.$
65. $t : h,\ c,\ \gamma.$
66. $t : h,\ r,\ \alpha.$
67. $t : h,\ \rho,\ \beta.$
68. $t : h,\ p - q,\ \alpha.$
69. $t : h,\ a - b,\ \gamma.$
70. $a : b,\ \alpha - \beta,\ h.$
71. $a : b,\ \alpha - \beta,\ r.$
72. $a : b,\ \alpha - \beta,\ \rho.$
73. $a : (p - q),\ c,\ \beta.$
74. $a : (p - q),\ b + c,\ \beta.$
75. $a + b : p - q,\ c,\ \beta.$

76. $a+b : p-q,\ h,\ \beta.$

77. $p : q,\ h,\ \alpha.$

78. $p : q,\ h,\ \gamma.$

79. $p : q,\ \rho,\ \gamma.$

80. $p : q,\ b+c,\ \alpha.$

81. $p : q,\ \alpha-\beta,\ h.$

82. $p : q,\ h,\ t.$

83. $p : q,\ \alpha,\ r.$

84. $h : m,\ u-v,\ \alpha.$

85. $h : m,\ \alpha-\beta,\ t.$

86. $h : m,\ a+b,\ \gamma.$

87. $c : h,\ \alpha-\beta,\ a+b.$

88. $c : h,\ \alpha-\beta,\ a-b.$

To construct a rectangle, given :

89. $a : f,\ b+f.$

90. $a : f,\ a+b.$

91. $a : f,\ a+b-f.$

92. $a : b, f.$

93. $a : b, f+a.$

94. $a : b, f-a.$

To construct a parallellogram, given :

95. $a : b,\ \alpha,\ h.$

96. $a : b,\ \alpha,\ a+h.$

97. $a \cdot b,\ \alpha,\ a-h.$

98. $a : b,\ \alpha, f+g.$

99. $a : b,\ \alpha,\ a+f.$

100. $a : b,\ \alpha,\ h+f.$

To construct a chord quadrilateral, given $a : b$, and also,

101. $r,\ c,\ \alpha.$

102. $c,\ d,\ \alpha.$

103. $c,\ g,\ \alpha.$

104. $r, f,\ \alpha.$

105. $c, f,\ \alpha.$

106. $f,\ g,\ \alpha.$

107. $r,\ \alpha,\ \beta.$

108. $c,\ \alpha,\ \beta.$

109. $f,\ \alpha,\ \beta.$

To construct a chord quadrilateral, given :

110. $a,\ c,\ \alpha,\ \theta.$

111. $a,\ c,\ d,\ \theta.$

112. $a,\ c, f,\ \theta.$

To construct a tangent quadrilateral, given :

113. $f,\ \alpha,\ \beta,\ \gamma.$

114. $a,\ b,\ c : d,\ \alpha.$

115. $a : b,\ \rho,\ \alpha,\ \gamma.$

116. $a : b,\ c,\ \alpha,\ \gamma.$

To construct a quadrilateral, given the four angles which the diagonal f forms with the sides, and also

117. $a+b.$ **118.** $g.$ **119.** $f+g.$ **120.** $a+b+c+d.$

§ 18. The Problem of Apollonius.

To construct a circle which shall touch three given circles.

This problem was first solved by Apollonius, a famous Greek geometer, who lived about 200 B.C. It is the last of the following series of ten closely related problems.

In these problems P, P_1, P_2 denote given points; L, L_1, L_2 given lines; K, K_1, K_2 given circles; O, O_1, O_2 their centres; r, r_1, r_2 their radii; X the centre of the required circle.

1. To construct a circle through P, P_1, P_2.

Solution in all text-books. When is the problem impossible?

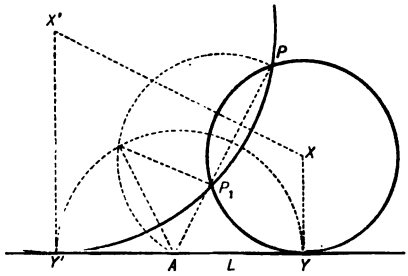


Fig. 82.

2. To construct a circle through P, P_1, and touching L.

Analysis. Let the line PP_1 (*Fig.* 82) cut L in A, and the required circle cut L in Y. Then $AY^2 = AP \times AP_1$, whence Y may be found. One locus of X is the line through Y and $\perp$ to AY, and a second locus is the bisector of the angle PAY.

Discussion. Since AY may be taken from A upon L in two directions, there are in general two solutions.

Special case: $PP_1 \parallel$ to L.

3. To construct a circle through P, P_1, and touching K.

Analysis. Let the required circle touch K in Y (*Fig.* 83), and the common tangent through Y meet PP_1 in M, and MAB be any secant through M, cutting K in A and B; then

$$MY^2 = MP \times MP_1 = MA \times MB. \qquad \text{(No. 166.)}$$

Hence, a circle may be described through P, P_1, A, B (Ex. 10, § 13).

Conversely, to determine Y, describe any circle through P and P_1, let it cut K in A and B, produce AB to meet PP_1 in M, and through M draw a tangent to K. When Y is known, the problem is reduced to Ex. 1. *Fig.* 83 shows the construction.

Discussion. The two tangents through M to K furnish, in general, two solutions. Special cases: (i.) P without K, and P_1 in the tangent from P to K; (ii.) P in the circumference of K and P_1, (*a*) in, (*b*) within, (*c*) without, the circumference of K; (iii.) P and P_1 within K; (iv.) P, P_1, and O in a straight line; (v.) P and P_1 equidistant from K.

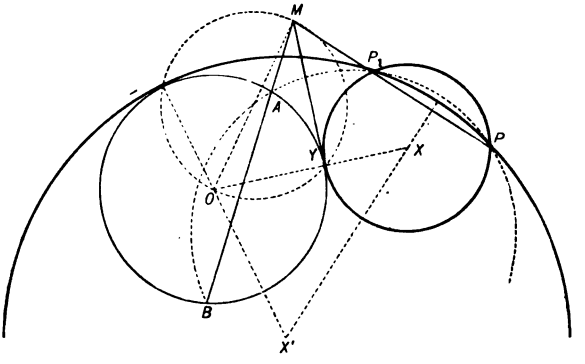


Fig. 83.

4. To construct a circle through P, and touching L, L_1.

Analysis. Bisect that angle formed by L and L_1 (*Fig.* 84), in which P lies; draw a perpendicular PB from P to the bisector; produce PB to Q, making $BQ = BP$; then Q is a second point in the required circumference, and the problem is reduced to **Ex. 2.**

Second Analysis. About any point O (*Fig.* 84), in the bisector as centre, describe a circle touching L and L_1; the point A is a centre of similitude of this circle and the required circle. (Ex. 36, § 13.) Let AP cut this auxiliary circle in D and E; then OD or OE is parallel to the radius of the required circle drawn to P. (Ex. 33, § 13.)

Discussion. In general, there are two solutions. Special cases: (i.) $L \parallel$ to L_1; (ii.) P lies in L; (iii.) P lies in the bisector AO.

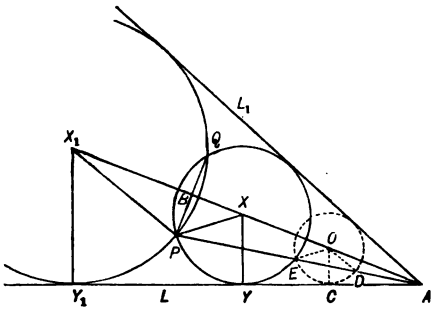


Fig. 84.

5. To construct a circle through P, and touching L, K.

Analysis. Let the required circle touch L in Y (*Fig.* 85), and K in Z. Produce YZ to meet K in A, join AO, and produce it to meet K a second time in B and L in C. Join BZ and AP, and let AP meet the required circle a second time in Q.

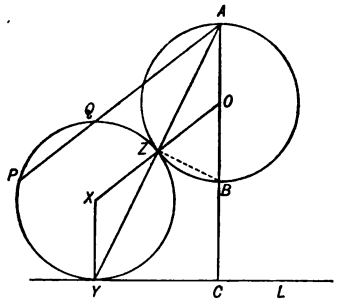


Fig. 85.

$\angle XYZ = \angle XZY = \angle AZO = \angle OAZ$. $\therefore AC$ is $\parallel$ to XY. But XY is $\perp$ to L; $\therefore AC$ is $\perp$ to L. Also, BZ is $\perp$ to AY; $\therefore$ the $\triangle ACY$, ABZ are similar; whence, $AB \times AC = AZ \times AY$. Also, $AQ \times AP = AZ \times AY$ (No. 165). Hence, $AB \times AC = AQ \times AP$. $\therefore$ the circumference which passes through B, C, and P must also pass through Q. (**Ex.** 10, § 13.) By describing this circumference, Q is found, and the problem is then reduced to **Ex.** 2.

Discussion. If L does not cut K, and P lies without K, as shown in *Fig.* 85, there are four solutions, two of which are obtained by joining P to A, and the other two by joining P to B. There are numerous special cases for different relative positions of P, L, and K.

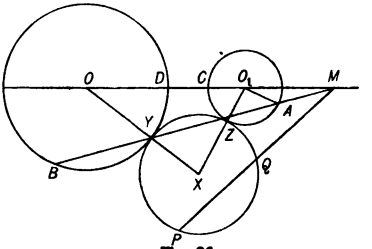


Fig. 86.

6. To construct a circle through P, and touching K, K_1.

Analysis. Let the required circle touch K, K_1 in Y, Z (*Fig.* 86). Produce YZ to meet OO_1 in M. $\angle OYB = \angle XYZ = \angle XZY = \angle AZO_1 = \angle O_1AZ$, $\therefore$ OY is $\parallel$ to O_1A, $\therefore$ M is the direct centre of similitude of K and K_1 (Ex. 32, § 13), $\therefore$ $MD \times MC = MY \times MZ$ (Ex. 35, § 13). Join PM, cutting the required circle a second time in Q; then $MP \times MQ = MY \times MZ$ (No. 165), $\therefore$ $MP \times MQ = MD \times MC$, whence it follows that Q lies in the circumference passing through the three known points P, D, C. By describing this circumference and joining P to M, the direct centre of similitude of K and K_1, the point Q is found, and the problem will then be reduced to Ex. 3.

Discussion. The maximum number of solutions is four. In discussing the numerous special cases, the position of P relatively to the common tangents of K and K_1 must be considered.

7. To construct a circle touching L, L_1, L_2.

See Ex. 66, § 7. According to the relative positions of the lines the number of solutions is four, two, or none.

8. To construct a circle touching L, L_1, K.

Analysis. Suppose the problem solved, and a concentric circle described with X as centre and XO as radius (*Fig.* 87); this circle must touch parallels to L and L_1, respectively, drawn at the distance r from L and L_1. Therefore X is found by constructing this concentric circle, as explained in Ex. 4.

Discussion. The maximum number of solutions is 8. In discussing special cases, the positions of L and L_1 relatively to K must be considered.

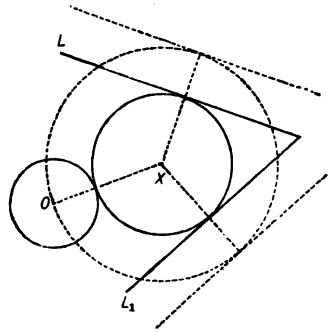


Fig. 87.

9. To construct a circle touching L, K, K_1.

Analysis. Suppose the required circle constructed, and also a concentric circle with X as centre and XO as radius; this circle must touch: (i.) a parallel to L, drawn at the distance r from L; (ii.) a circle with O_1 as centre and $r_1 - r$ as radius. Hence, X is found by constructing this concentric circle, as explained in Ex. 5.

Discussion. The use of $r_1 - r$ as radius of the auxiliary circle may give four solutions, and the use of $r_1 + r$ four more solutions.

10. To construct a circle touching K, K_1, K_2.

Analysis. A concentric circle, with XO as radius, must touch the circle with O_1 as centre and $r_1 - r$ as radius, and also touch the circle with O_2 as centre and $r_2 - r$ as radius. By constructing this concentric circle, therefore, as explained in Ex. 6, X will be determined.

Discussion. By using for the radii of the two auxiliary circles about O_1 and O_2 as centres different combinations of the four values, $r_1 - r$, $r_1 + r$, $r_2 - r$, $r_2 + r$, the maximum number of solutions is eight.

CHAPTER IV.

EQUIVALENT FIGURES.

§ 19. THEOREMS.

1. All parallelograms having equal bases, and contained between two parallel lines, are equivalent.

2. Two triangles having two sides equal, each to each, and the included angles supplementary, are equivalent.

3. Two parallelograms having two adjacent sides equal, each to each, and the included angles supplementary, are equivalent.

4. Every straight line drawn through the centre of a parallelogram divides it into two equivalent parts. What kind of figures are the two parts?

5. Every straight line drawn through the middle point of the median of a trapezoid, and cutting the two bases, divides the trapezoid into two equivalent parts. Is the theorem also true if the line cuts the legs instead of the bases?

6. In every trapezoid the triangle which has for base one of the legs of the trapezoid, and for vertex the middle point of the opposite side, is equivalent to one-half of the trapezoid.

7. The sum of the areas of two opposite triangles, formed by joining a point within a parallelogram to the four vertices, is equal to one-half of the area of the parallelogram.

8. The area of a trapezoid is equal to the product of one of its legs, and the distance of this leg from the middle point of the opposite side.

9. The triangle whose vertices are the middle points of the sides of a given triangle is equivalent to one-fourth of the given triangle.

10. The figure whose vertices are the middle points of the sides of any quadrilateral is a parallelogram, and is equivalent to one-half of the quadrilateral.

11. Mutually equiangular parallelograms have the same ratio as the products of two adjacent sides.

12. Similar polygons are to each other as the squares of two homologous diagonals.

13. If ABC is a right triangle, C the vertex of the right angle, BD a line cutting AC in D, then $BD^2 + AC^2 = AB^2 + DC^2$.

14. If upon the sides of a right triangle, as homologous sides, any three similar figures are constructed, the figure constructed upon the hypotenuse is equivalent to the sum of the figures constructed upon the legs.

15. The square constructed upon the sum of two lines is equivalent to the sum of the squares constructed upon the lines plus twice the rectangle having the lines for its dimensions.

16. The square constructed upon the difference of two lines is equivalent to the sum of the squares constructed upon the lines minus twice the rectangle having the lines for its dimensions.

17. The difference between the squares constructed upon two lines is equivalent to the rectangle having the sum and the difference of the two lines for its dimensions.

18. If p denote half the perimeter of a triangle, ρ the radius of the inscribed circle, prove that

$$\text{Area of a triangle} = \rho p.$$

19. If a, b, c denote the sides of a triangle, r the radius of the circumscribed circle, prove that

$$\text{Area of a triangle} = \frac{abc}{4r}.$$

20. If a, b, c denote the sides of a triangle, and $p = \frac{1}{2}(a+b+c)$, prove that

$$\text{Area of a triangle} = \sqrt{p(p-a)(p-b)(p-c)}.$$

21. Of all triangles which can be formed with two given lines for sides, that is the maximum in which the given lines form a right angle.

22. Of all the triangles which can be formed by the two sides of an angle and lines drawn through a fixed point in the bisector of the angle, the isosceles triangle is the minimum.

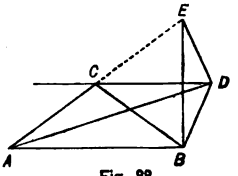


Fig. 88.

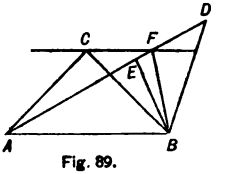


Fig. 89.

23. Of all the triangles which can be constructed upon a given line as base, and which have a given angle opposite this base, the isosceles triangle is the maximum.

Construct upon the given base a segment capable of containing the given angle.

24. Of all equivalent triangles standing upon the same base, the isosceles triangle has the minimum perimeter.

Of all equivalent triangles, which has the minimum perimeter?

Analysis. (*Fig.* 88.) Let ABC be an isosceles triangle, and ADB any other triangle, equal in area and having the same base AB. Then CD is ∥ to AB. Draw $BE \perp$ to CD, produce AC to meet BE in E, join DE, and show that $AC + CB < AD + DB$.

25. Of all triangles upon the same base, and having equal perimeters, the isosceles triangle is the maximum.

Analysis. Let AB (*Fig.* 89) be the given base, ABC the isosceles triangle, and ABD any other triangle having the same perimeter. Take $AE = 2\,AC$, draw $CF \parallel AB$, and show that D must be between CF and AB.

§ 20. Numerical Exercises.

Find the area of a square, given:

1. One side 160 yards.

2. One side 1.5^{m}.

3. Perimeter 25 feet.

4. Diagonal 36 yards.

5. Find in yards the side of a square field which contains 10 acres.

6. A square field contains 15$\frac{1}{2}$ acres; find the length of the fence which encloses it.

7. A square plot of ground, the side of which is 127 yards, has a path 1 yard wide round the *inside* of its perimeter. What will be the expense of gravelling the path at 12 cents per square yard?

8. What will it cost, at 18 cents per square foot, to tile a walk 2 feet 6 inches wide round the *outside* of a square court which is 30 yards long?

9. A square field 210 yards long has a path round the *inside* of its perimeter which occupies just one-seventh of the whole field. Find the width of the path.

Find the area of a rectangle, given:

10. Length 7 feet 6 inches, and breadth 6 feet 7 inches.

11. Length 210 yards, and perimeter 760 yards.

12. Length 108 yards, and diagonal 135 yards.

13. A street $1\frac{1}{4}$ miles long contains 5 acres. How wide is the street?

14. How many yards of paper 27 inches wide will be required to paper a room 18 feet long, 12 feet wide, and 11 feet high?

15. The perimeter of a rectangle is 72 feet, and its length is equal to twice its width; find its area.

16. A chain 80 feet long encloses a rectangle 15 feet wide. How much more surface would it enclose if the figure were a square?

17. The perimeter of a square, and also of a rectangle whose length is four times its breadth, is 100 yards. What is the difference in their areas?

18. A rectangle whose length is 25^m is equivalent to a square whose side is 15^m. Which has the greater perimeter, and by how much?

Find the dimensions of a rectangle, given:

19. Area 216 square feet, perimeter 60 feet.

20. Area 600 square feet, difference of sides 10 feet.

21. Area 60 square feet, diagonal 13 feet.

22. Area 756 square feet, sides are as 7 : 3.

23. Area 192 square feet, diagonal $\frac{5}{3}$ of one side.

24. Find the area of a rectangle if the diagonal is 75^m, and the sides are as 3 : 4.

25. A rectangle contains 2400 square feet, and the length is 20 feet more than the breadth. Find its dimensions.

26. The dimensions of a rectangle are 36 feet by 20 feet. If the length is diminished by 6 feet, how much must be added to the breadth, in order that the area may remain the same?

27. How many tiles 9 inches long and 4 inches wide will be required to pave a path 8 feet wide surrounding a rectangular court 60 feet long and 36 feet wide?

28. A carpenter requires $3\frac{1}{2}$ square feet of plank, and he has a plank $1\frac{1}{4}$ feet wide from which to cut it off. Find the length of the piece which he must cut off.

29. A wall round a square field at \$0.96 per yard cost \$537.60. What would it have cost if the field had been in the shape of a rectangle with a length of 196 yards?

30. A man bought a corner lot in the shape of a rectangle 121 yards long and 100 yards wide, at \$5000 per acre. He laid out two streets directly through the middle of the lot, each street being 15 yards wide and parallel to two opposite sides of the lot. The cost of making the streets was \$2000. He then sold the remainder of the land at 20 cents per square foot. Did he gain or lose, and how much?

31. The perimeters of two rectangular lots are 102 yards and 108 yards, respectively. The first lot is eight-ninths as wide as it is long, and the second lot is twice as long as it is wide. The value per square foot of each lot is \$1.00. What is the difference in the value of the lots?

Find the area of a rhombus, given:

32. Perimeter 600 feet, altitude 75 feet.

33. Perimeter 40^{m}, shorter diagonal 12^{m}.

34. The two diagonals 18 feet and 12 feet, respectively.

35. Sum of diagonals 12 feet, their ratio 3 : 5.

36. A rhombus contains 100 square feet, and the shorter diagonal is 10 feet. Find the length of the other diagonal.

37. A rhombus and a square have equal perimeters. Which has the greater area? What is the ratio of their areas if the altitude of the rhombus is equal to one-half that of the square?

38. What ratio has the area of a rhombus whose acute angle is 60°, to that of a square having an equal perimeter?

39. A rhombus and a rectangle have equal bases and equal areas. Find their perimeters if one side of the rhombus is 5 feet, and the altitude of the rectangle is $3\frac{1}{2}$ feet.

40. The area of a rhombus is $60^{sq m}$, and the shorter diagonal is equal to one side of the rhombus. Find the perimeter.

41. The diagonals of a rhombus are 90 yards and 120 yards, respectively. Find the length of one side, and also the distance between the parallel sides.

42. Find the area of a parallelogram if the base is 40 feet 6 inches, and the altitude is 28 feet 9 inches.

43. If two parallelograms have equal areas, and the base of one is three times that of the other, what is the ratio of their altitudes?

44. What is the ratio of the areas of two parallelograms of given bases and altitudes?

45. What is the ratio of the areas of two triangles of given bases and altitudes?

46. If a, a' denote the bases, and h, h' the altitudes, respectively, of two equivalent triangles, state, in the form of a proportion, the relation of the four quantities a, a', h, h'.

47. A parallelogram and a triangle have equal altitudes, but the base of the parallelogram is equal to one-half that of the triangle. Compare their areas.

Find the area of a triangle, given :

48. Base 64 miles, altitude 14 miles.

49. Rt. $\triangle$, legs 6 feet and 10 feet.

50. Rt. isos. $\triangle$, one leg 18 feet.

51. Rt. isos. $\triangle$, hypotenuse 20 feet.

52. Rt. $\triangle$, one leg 5 feet, hypotenuse 13 feet.

53. Rt. $\triangle$, one leg 3 inches, opposite angle 30°.

54. Isos. $\triangle$, base 36 feet, one leg 30 feet.

55. Isos. $\triangle$, base 2 feet, angle at base 60°.

56. Equilat. $\triangle$, one side 40 feet.

57. Equilat. $\triangle$, altitude 24 feet.

58. The altitudes of two triangles are equal, and their bases are 12 feet and 16 feet, respectively. What is the ratio of their areas?

59. The altitudes of two triangles are equal, and their bases are 20 feet and 30 feet, respectively. What is the base of a triangle equivalent to their sum, and having an altitude one-fourth as great?

60. A house is 40 feet long, 30 feet wide, 25 feet high to the roof, and 35 feet high to the ridgepole. Find the number of square feet in the entire exterior surface.

61. What must be the length of the hypotenuse of a right triangle in order that its area may be 500^{sq}?

62. Find the area of a right triangle if the perimeter is 60 feet, and its sides are as 3 : 4 : 5.

63. Find the area of a right triangle if its sides are as 3 : 4 : 5, and the altitude upon the hypotenuse is 12 feet.

64. The legs of a right triangle are 30 feet and 40 feet. Find the areas of the parts into which the triangle is divided by a perpendicular drawn from the vertex of the right angle to the hypotenuse.

65. Find the area of a right triangle if one leg is 15^m, and the altitude upon the hypotenuse is 8^m.

66. The area of a right triangle is 300^{qm} and the hypotenuse is equal to 50^m. Find the legs.

67. The area of a triangle is 875 square feet. Find its base and its altitude if they are as 14 : 5.

68. ABC is a triangle, and AD the perpendicular from A upon BC. If $AD = 13$ feet, and the lengths of the perpendiculars from D to AB and AC are 5 feet and $10\frac{2}{3}$ feet, respectively, find the area of the triangle.

69. Find the area of a triangle if the three sides are 104 feet, 111 feet, and 175 feet, respectively.

70. How many square feet of carpet are required to cover a triangular floor whose sides measure, respectively, 26 feet, 35 feet, and 51 feet?

71. The two legs of a right triangle are 1 foot and 2 feet. Find the radius of the inscribed circle.

72. Given the sides a, b, c of a triangle; find the radii r, ρ of the circumscribed and inscribed circles.

73. The three sides of a triangle are: AB 100 feet, BC 89 feet, AC 21 feet. Find the length of the perpendicular from C to AB.

74. Find the area of a triangle if the perimeter is 14^m and the radius of the inscribed circle is 1.07^m.

Find the area of a trapezoid, given:

75. Bases 50 feet and 34 feet, altitude 25 feet.

76. Median 25 feet, altitude 12 feet.

77. The area of a trapezoid is 700 square feet, the bases are 30 feet and 40 feet, respectively. Find the distance between the bases.

78. A trapezoid contains 240 square feet, and its altitude is 16 feet. Find the two bases: (i.) if one is 3 feet longer than the other; (ii.) if they are in the ratio 2 : 3.

79. The value of a field in the shape of a trapezoid is $5800. The bases are 200 yards and 119 yards, and the distance between them is 110 yards. Find the value per acre.

80. The bases of a trapezoid are 32 feet and 20 feet. Each of the other sides is equal to 10 feet. Find the area of the trapezoid.

81. Find the area of a trapezoid if the altitude is equal to the median, the difference of the bases is 1 foot, and the greater base is equal to the hypotenuse of a right triangle whose legs are the smaller base and the altitude.

82. A lot of land has the shape of a trapezoid. Its bases are 100^m and 40^m. Each of the other sides is 50^m. Find (i.) the area in ars of the trapezoid; (ii.) the area of the triangle formed by producing the equal sides to their intersection.

83. Find the area of a trapezoid considered as the sum of a triangle and a parallelogram.

84. Find the area of a trapezoid considered as the difference of two triangles.

85. A lot of land has the shape of a trapezium. One diagonal is 108 feet, and perpendiculars upon it from the opposite vertices are 55 feet 3 inches, and 60 feet 9 inches, respectively. What will the lot cost at 60 cents per square yard?

86. $ABCD$ is a trapezium, having $AB = 87$ feet, $BC = 119$ feet, $CD = 41$ feet, $DA = 169$ feet, $AC = 200$ feet. Find the area.

87. Find the area of a tangent quadrilateral whose perimeter is equal to 400 feet, the radius of the circle being 25 feet.

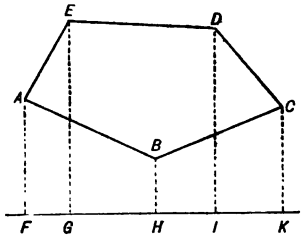


Fig. 90.

88. Find the area of the polygon $ABCDE$ (*Fig.* 90) if perpendiculars are dropped from the vertices to a line L in the plane of the figure, and the following lines are measured:

$AF = 400$ feet; $DI = 680$ feet; $GH = 250$ feet;
$BH = 200$ feet; $EG = 700$ feet; $HI = 200$ feet;
$CK = 380$ feet; $FG = 150$ feet; $IK = 220$ feet.

89. What is the side of a square equivalent to a rectangle 200 feet long and 32 feet wide?

90. The dimensions of a triangle are: base, 360 feet; altitude, 240 feet. Find the altitude of an equivalent parallelogram if its base is 270 feet.

91. A triangle whose base is 50 feet is equivalent to a rectangle 40 feet long and 25 feet wide. What is the height of the triangle?

92. Upon the diagonal of a rectangle 24 feet long and 10 feet wide a triangle is constructed equal in area to the rectangle. Find its altitude.

93. A triangle whose base is 15 feet and altitude 8 feet is transformed into a rhombus whose altitude is 6 feet. Find the perimeter of this rhombus.

94. If a triangle whose base is 22 feet and altitude 15 feet is transformed into a rhombus whose longer diagonal is 12 feet, what is the length of the other diagonal?

95. The bases of a trapezoid are 16 feet and 20 feet, the altitude is 12 feet. What is the base of an equivalent rectangle, having an equal altitude?

96. The length of a rectangle is 60 feet, the breadth 40 feet. If the length is reduced 5 feet, how much must be added to the breadth in order that the area may remain the same?

97. The legs of a right triangle are each 18 feet long. Find the side of an equivalent square.

98. Find the side of a square equivalent to an equilateral triangle whose side is 12 feet.

99. Find the side of an equilateral triangle equivalent to a square whose side is 12 feet.

100. Find the side of an equilateral triangle equivalent to the triangle whose sides are 12 feet, 16 feet, and 18 feet.

101. Find the side of a square equivalent to a quadrilateral whose sides are 6 feet, 8 feet, 10 feet, and 12 feet, and one of whose diagonals is 14 feet.

102. Find the side of a rhombus composed of two equilateral triangles, and equivalent to another rhombus whose diagonals are 12 feet and 16 feet.

103. The perimeter of an isosceles triangle, in which the base is half the length of one leg, is 120 feet. Find the side of a square equal in area to the triangle.

104. The sides of a triangular field are 65 yards, 119 yards, and 138 yards, respectively. Find the side of a square field containing the same amount of land.

105. The perimeter of a polygon circumscribed about a circle is 320 feet; the radius of the circle is 40 feet. Find the side of the square equivalent to this polygon.

106. A triangular lot of land containing 600 square feet, and having a base of 75 feet, is to be divided into two parts containing 240 square feet and 360 square feet, respectively. If the line of division starts from the opposite vertex, how will it divide the base?

107. A triangle containing 80 square feet is divided by a line drawn through one vertex into two parts, one of which contains 16 square feet more than the other. If the base is 20 feet, into what two parts is it divided by the line of division?

108. A line of division is drawn between two sides of a triangle, dividing it into a triangle and a quadrilateral. What parts are these two figures, respectively, of the entire triangle, if the line of division cuts off the following parts of the two sides, reckoned from the intersection of the sides? (i.) $\frac{1}{2}$ and $\frac{1}{3}$; (ii.) $\frac{2}{3}$ and $\frac{3}{4}$; (iii.) $\frac{1}{2}$ and $\frac{1}{2}$; (iv.) $\frac{1}{3}$ and $\frac{1}{3}$; (v.) $\frac{1}{m}$ and $\frac{1}{n}$; (vi.) $\frac{1}{n}$ and $\frac{1}{n}$.

109. What parts of two sides of a triangle, reckoned from their intersections, must be taken, in order that the portion of the triangle contained between the two parts may be two-fifths of the entire triangle?

110. In a triangle ABC, $AB = 20$ feet, $AC = 64$ feet. Through a point D in AB, 8 feet from A, a line DE is to be drawn, meeting AC in E, so that the triangle ADE shall be three-eighths of the entire triangle. Find AE.

111. How must two lines be drawn through the middle point of one side of a triangle in order that the triangle may be divided into parts which shall be to each other as $1:2:3$?

112. A triangular field ABC is to be divided into three parts which shall be as $2:5:3$, by lines starting from a point D in AB, 90 yards from A and 150 yards from B. If E, F are the points in AC, BC to which the lines should be run, find AE and BF in terms of AC and BC, respectively.

Through a point D in the side AB of the triangle ABC a line DE is drawn parallel to BC. Find the value of AD in terms of AB if:

113. DE divides triangle ABC into two equivalent parts.

114. Triangle $ADE = \frac{1}{3}$ triangle ABC.

115. Trapezoid $BDEC = \frac{1}{4}$ triangle ABC.

116. Triangle ADE: trapezoid $BDEC = m:n$.

117. The base AB of a triangle is 40 feet in length. The foot D of the altitude is distant 30 feet from A; to find a point E in AB such that the perpendicular erected at E will divide the triangle into two parts having the ratio $3:5$.

118. What part of a parallelogram is the triangle cut off by a line drawn from one vertex to the middle point of one of the opposite sides?

119. The area of a parallelogram is 60 square feet. In what ratio must a line through one vertex divide the opposite side, in order that the triangle cut off may contain 25 square feet?

120. What part of a parallelogram is contained between one-half of one side and two-thirds of the opposite side?

121. A parallelogram is divided into two equivalent trapezoids by a line which divides one side in the ratio 1 : 2. In what ratio does this line divide the opposite side?

122. In the trapezoid $ABCD$ it is required to draw through the point E, in the upper base DC, a line EF which shall divide the trapezoid into two equivalent parts. Find AF, given $AB = 6$ feet, $DC = 4\frac{1}{2}$ feet, $DE = 1\frac{1}{2}$ feet.

123. In the trapezoid $ABCD$ the bases are $AB = 5$ feet, $DC = 3$ feet. To determine a point X in the diagonal DB, such that a line through X, and parallel to AD, will divide the trapezoid into two parts having the ratio 2 : 3.

124. A trapezoid whose bases are 12 feet and 7 feet, respectively, and whose altitude is denoted by h, is divided into two equivalent parts by a line parallel to the bases. Find the length of this line and the altitude of the upper part.

125. In a trapezoid, given the two bases a, b, and the altitude h. The legs are divided each into three equal parts by lines parallel to the bases. Find, in terms of a, b, and h, the areas of the three parts into which the trapezoid is divided.

126. In two similar polygons two homologous sides are 15 feet and 25 feet. The area of the first polygon is 450 square feet; what is the area of the other polygon?

127. The base of a triangle is 32 feet, and its altitude 20 feet. What is the area of the triangle formed by drawing a line parallel to the base and 5 feet from the vertex?

128. The altitude of a triangle is 6 feet. What is the homologous altitude of a similar triangle 100 times as large?

129. If one side of a triangle is a, what is the value of the homologous side of a similar triangle one-half as large?

130. The side of a square is 4 feet. Find the side of a square which shall be to the given square as 3 : 5.

131. The side of a square is 4 feet. Find the side of a square $3\frac{1}{2}$ times as large.

132. The diagonals of two squares are 12 feet and 18 feet. Find the diagonal of a square equivalent to their sum.

133. The sides of two equilateral triangles are 3 feet and 4 feet. What is the side of the equilateral triangle equivalent to their sum?

134. The homologous sides of two similar hexagons are 33 feet and 56 feet. Find the homologous side of a similar hexagon equivalent to their sum.

135. What is the ratio of the square constructed upon the diagonal of a given square to the given square?

136. What is the ratio of the square constructed upon a given line, and the isosceles right triangle constructed upon this line as (i.) hypotenuse; (ii.) leg?

137. If the side of one equilateral triangle is equal to the altitude of another, find the ratio of their areas.

138. The sides of a triangle are 10 feet, 17 feet, and 21 feet. Find the areas of the two parts into which the triangle is divided by the bisector of the angle formed by the first two sides.

139. In a trapezoid one base is 10 feet, the altitude is 4 feet, and the area is 32 square feet. Find the length of a line drawn between the legs parallel to the given base and distant 1 foot from it.

140. Find the sides of an isosceles trapezoid circumscribed about a circle, knowing that the sum of the bases is $2a$, and that the area is equal to that of a square whose side is b. Examine the case in which a is equal to b.

Find the area F of a triangle, given:

141. Rt. $\triangle$, a, c.

142. Rt. $\triangle$, a, h.

143. Rt. $\triangle$, q, h.

144. Rt. $\triangle$, p, q.

145. Equilat. $\triangle$, a.

146. Equilat. $\triangle$, h.

147. Isos. $\triangle$, a, c.

148. Isos. $\triangle$, a, h.

149. Isos. $\triangle$, h, h_a.

150. In an equilateral triangle given F; find a and h.

151. If the altitude h of a triangle is increased by the length m, by how much must the base a be diminished in order that the area may remain the same?

152. Find the change in the area of a triangle of base a and altitude b in the following cases:

(i.) if a and h are increased by m and n, respectively;

(ii.) if a and h are diminished by m and n, respectively;

(iii.) if a is increased by m, and h diminished by n;

(iv.) if a is diminished by m, and h increased by n.

153. Find the area F of a square, given (i.) the sum s of a diagonal and a side; (ii.) the difference d of a diagonal and a side.

154. The distances of a point within a rectangle from the sides a, b are m, n, respectively. Through this point parallels to the sides are drawn. Find the areas of the parts into which the rectangle is divided.

155. Find the diagonal f of a parallelogram, given the sides a, b, and the other diagonal g.

156. The sides of a rectangle are a, b. Find the length of a line which shall have the same ratio to the perimeter of the rectangle as its square has to the area of the rectangle.

157. Find the sides of a rectangle, given the diagonal f and the area F.

158. Find the diagonals of a rhombus, given the side a and the area F.

159. In the trapezoid $ABCD$, given the bases $AB = a$, $CD = c$. Find the lengths x, y of lines parallel to the bases and dividing the trapezoid into parts which, counted from CD, are to one another as $m : n : p$.

§ 21. Problems.

For the notation in the triangle ABC, see page 44; also, l and k are used to denote given lengths, ϕ a given angle.

To construct a triangle equivalent to:

1. The sum of two given triangles with equal altitudes.
2. The difference of two given triangles with equal altitudes.
3. The sum of two given triangles with equal bases.
4. The difference of two given triangles with equal bases.
5. n times as large as a given triangle.

To transform the triangle ABC into:

6. A triangle with c unchanged and l instead of a.

7. A triangle with c unchanged and ϕ instead of a.

8. A triangle with c unchanged and l instead of h_a.

9. A right triangle with c for one leg.

10. A right triangle with c for hypotenuse.

11. An isosceles triangle with c for base.

12. An isosceles triangle with c for one leg.

13. A rectangle with c for one side.

14. A parallelogram with c and a unchanged.

15. A parallelogram with a and b for diagonals.

16. A rhombus with a for one diagonal.

17. To transform the triangle ABC into a triangle with l instead of c, and a unchanged.

Analysis. Upon AB (produced if necessary) take $AD = l$, draw $BE \parallel$ to CD and meeting AC (produced if necessary) in E; then $\triangle ADE \backsim \triangle ABC$.

To transform the triangle ABC into:

18. A triangle with l instead of c, and ϕ instead of a.

19. A triangle with l instead of c, and k instead of b.

20. A triangle with l instead of c, and k instead of h_a.

21. A triangle with l instead of c, and k instead of m.

22. A right triangle with l for one leg.

23. A right triangle with l for hypotenuse.

24. An isosceles triangle with l for the base.

25. An isosceles triangle with l for one leg.

26. A triangle with l instead of c, and ϕ instead of γ.

27. A triangle with l instead of r, and ϕ instead of γ.

28. A triangle with l instead of c, and k instead of r.

29. To transform the triangle ABC into a triangle with l instead of h, and a unchanged.

Analysis. Draw $AD \perp$ to AB and equal to l, $DE \parallel$ to AB, and meeting AC (produced if necessary) in E, $CF \parallel$ to EB and meeting AB (produced if necessary) in F; then $\triangle AEF \backsim \triangle ABC$.

To transform the triangle ABC into:

30. A triangle with l instead of h, and ϕ instead of a.

31. A triangle with l instead of h, and k instead of m.

32. A triangle with l instead of h, and k instead of h_a.

33. A right triangle with l for altitude upon hypotenuse.

34. An isosceles triangle with l for altitude upon the base.

35. A triangle with l instead of h, and ϕ instead of γ.

36. A triangle with l instead of h, and k instead of r.

37. A triangle with l instead of h, and ϕ instead of $\angle am$.

38. A triangle with one vertex in a given point.

To construct a triangle equivalent to:

39. The sum of two given triangles.

40. The difference of two given triangles.

To transform the triangle ABC into:

41. A triangle with base and altitude equal.

42. A triangle with base and altitude equal, and l instead of a.

43. A triangle with base and altitude equal, and l instead of m.

44. A triangle with base and altitude equal, and ϕ instead of γ.

45. To transform the triangle ABC into a triangle similar to a given triangle PQR.

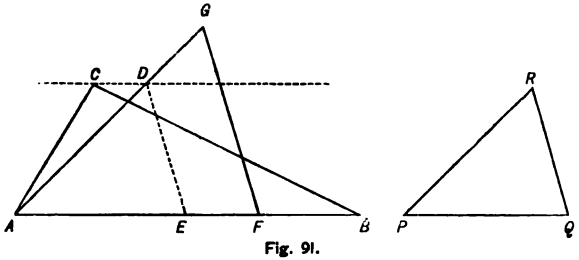


Fig. 91.

Analysis. Let AFG (*Fig.* 91) be the required triangle constructed with the side AF in the line AB. Draw $CD \parallel$ to AB and meeting AG in D, and $DE \parallel$ to GF; then, by similar triangles,

$$\triangle AED : \triangle AFG = AE^2 : AF^2;$$

or, since $\triangle AFG \backsim \triangle ABC$,

$$\triangle AED : \triangle ABC = AE^2 : AF^2.$$

Since the triangles AED and ABC have equal altitudes,

$$\triangle AED : \triangle ABC = AE : AB.$$

$$\therefore AE^2 : AF^2 = AE : AB,$$

and

$$AF^2 = AB \times AE.$$

That is, AF is the mean proportional between the two known lines AB and AE.

To transform the triangle ABC into:

46. A right triangle with ϕ for one of the acute angles.

47. An isosceles triangle with ϕ for the angle at the vertex.

48. An equilateral triangle.

49. A triangle with one angle unchanged and the opposite side parallel to a given line.

50. To transform the triangle ABC into a square.

51. To transform the triangle ABC into a trapezoid with c for one base and for one adjacent angle, and ϕ for the other adjacent angle.

To transform the triangle ABC into:

52. A triangle with a given perimeter.

53. A right triangle with a given perimeter.

54. A right triangle with l for radius of inscribed circle.

55. To transform a right triangle into an isosceles triangle having an angle in common with the right triangle.

To transform a parallelogram into:

56. A parallelogram having a given side l.

57. A parallelogram having a given angle ϕ.

58. A parallelogram having a given altitude l.

To transform a square into:

59. A right triangle. 60. An isosceles triangle.

61. An equilateral triangle.

62. A rectangle having a given side.

63. A rectangle having a given perimeter.

64. A rectangle having a given difference of its sides.

65. A rectangle having a given diagonal.

To transform a rectangle into:

66. A square. 67. An equilateral triangle.

68. A rectangle having a given side.

69. A rectangle having a given perimeter.

70. A rectangle having a given difference of its sides.

71. A rectangle having a given diagonal.

To construct a parallelogram equivalent to the:

72. Sum of two given parallelograms of equal altitudes.

73. Difference of two given parallelograms of equal altitudes.

74. Sum of two given parallelograms of equal bases.

75. Difference of two given parallelograms of equal bases.

76. Sum of two given parallelograms.

77. Difference of two given parallelograms.

To transform a parallelogram into:

78. A triangle.

79. A right triangle.

80. An isosceles triangle.

81. An equilateral triangle.

82. A square.

83. A rhombus having for diagonal one side of the parallelogram.

84. A rhombus having a given diagonal.

85. A rhombus having a given side.

86. A rhombus having a given altitude.

87. A parallelogram having given diagonals.

88. A parallelogram having a given side and a given diagonal.

89. To transform a rhombus into a square.

90. To inscribe in a given circle a rectangle equivalent to a given square.

To transform a trapezoid into:

91. A triangle.

92. A square.

93. A parallelogram having for base one base of the trapezoid.

94. An isosceles trapezoid.

To transform a trapezium into:

95. A triangle.

96. An isosceles triangle having a given base.

97. A parallelogram.

98. A trapezoid with one side and the two adjacent angles unchanged.

99. To transform a given polygon M into a polygon similar to a given polygon N.

Hint. First transform M and N into squares.

100. Given an angle BAC and a point P in AB; through P to draw a line cutting AC in Q so that the triangle APQ shall be equivalent to a given square.

101. Through a given point P to draw a line cutting the sides of a given angle BAC in X and Y so that the triangle AXY shall be equivalent to a given triangle DEF.

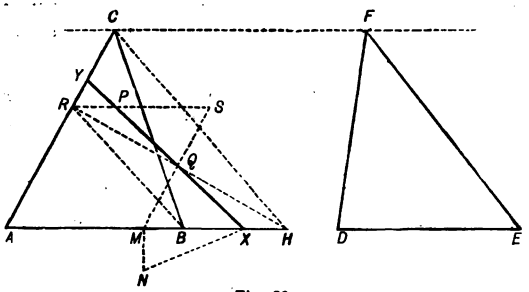


Fig. 92.

Transform the given $\triangle DEF$ (*Fig.* 92) into the $\triangle ABC$ having the given angle for one of its angles. Draw $PR \parallel$ to AB, and transform the $\triangle ABC$ into the $\triangle ARH$, having R for vertex, and the base AH in the line AB. Then transform the $\triangle ARH$ into the parallelogram $ARSM$, having the side AR and the angle A in common with the $\triangle ARH$. Since $\triangle AXY$ must be equal to this parallelogram, $\triangle PQS = \triangle PRY + \triangle QMX$. Since these three triangles are similar,

$$\frac{\overline{PR}^2}{\overline{PS}^2} = \frac{\triangle PRY}{\triangle PQS}, \text{ and } \frac{\overline{MX}^2}{\overline{PS}^2} = \frac{\triangle MQY}{\triangle PQS};$$

whence $\overline{PR}^2 + \overline{MX}^2 = \overline{PS}^2$.

Therefore, the point X is found by constructing the rt. $\triangle MNX$, in which MN is $\perp$ to AB and equal to PR, and $NX = PS$.

102. To construct a square which shall be to a given square as 5 : 3.

To construct a square equivalent to:

103. One-half of a given square.

104. One-third of a given square.

105. Five-eighths of a given square.

106. Nine-sevenths of a given square.

107. Three-fifths of a given pentagon.

108. The sum of two given equilateral triangles.

109. The sum of a given triangle and a given rectangle.

110. To construct upon a given line as base a triangle equivalent to the difference between a given rectangle and a given trapezoid.

111. To construct a polygon equivalent to the sum of two given polygons M, N, and similar to a third given polygon R.

112. To construct a polygon equivalent to the difference of two given polygons M, N, and similar to a third polygon R.

To divide into n equivalent parts (*e.g.*, $n = 5$):

113. A triangle. **115.** A trapezoid.

114. A parallelogram. **116.** A trapezium.

Analysis for Ex. 116. Divide a diagonal into n equal parts, and join the points of division to the opposite vertices.

To divide into two parts having the ratio $m : n$; (*e.g.*, $m = 2$, $n = 3$):

117. A triangle. **119.** A trapezoid.

118. A parallelogram. **120.** A trapezium.

To divide into three parts which shall be to one another as $m : n : p$; (*e.g.*, $m = 1$, $n = 2$, $p = 3$):

121. A triangle. **122.** A parallelogram.

123. A trapezoid. **124.** A trapezium.

125. To divide the triangle ABC into five equivalent parts by a broken line passing from C to c, from c to a, from a to c, and from c to a.

To divide the triangle ABC, by lines through a given point P in AB, into:

126. Two equivalent parts.

127. Three equivalent parts.

128. Five equivalent parts.

129. Two parts in the ratio 3 : 4.

Analysis for Ex. 126. If PQ is the required line, M the middle point of AB, then $\triangle PCQ = \triangle PCM$; $\therefore MQ$ is $\parallel$ to CP.

130. To cut from the triangle ABC a triangle equivalent to a given triangle by a line of division drawn from a point P in AC.

Hint. Transform the given triangle into another with CP for one side and ACB for an adjacent angle.

131. To find a point within a triangle such that the lines joining this point to the vertices shall divide the triangle into three equivalent parts.

Hint. Draw the medians.

132. To find a point within a triangle such that the lines joining this point to the vertices shall divide the triangle into parts proportional to the numbers m, n, p; (*e.g.*, $m=1$, $n=2$, $p=3$).

133. To divide a triangle into five equivalent parts by lines drawn from a given point within the triangle.

To divide a triangle by lines parallel to one side into:

134. Two equivalent parts.

135. Three equivalent parts.

136. Two parts in the ratio 3 : 5.

137. Parts proportional to 1, 2, 3.

138. To divide a triangle into two equivalent parts by a line perpendicular to one side.

139. To divide a triangle into three equivalent parts by lines parallel to one of the medians.

140. To divide a triangle into four equivalent parts by lines parallel to one of the bisectors.

141. To divide a triangle into two equivalent parts by a line drawn through a given point.

142. To divide a triangle into two parts in the ratio 2 : 5 by a line drawn through a given point.

To divide by lines drawn from one vertex:

143. A parallelogram into three equivalent parts.

144. A parallelogram into four equivalent parts.

145. A parallelogram into five equivalent parts.

146. A parallelogram into two parts in the ratio 3 : 4.

147. A trapezoid into two equivalent parts.

148. A trapezoid into three equivalent parts.

149. A trapezium into two equivalent parts.

150. A trapezium into two parts in the ratio 1 : 2.

151. An octagon into two equivalent parts.

To divide a parallelogram by lines drawn from a given point in one of the sides into:

152. Two equivalent parts. **153.** Five equivalent parts.

154. Two parts in the ratio 3 : 5.

To divide a parallelogram into two equivalent parts by a line:

155. Drawn through a given point P.

156. Parallel to a given line L.

To divide a trapezoid into two equivalent parts by a line:

157. Parallel to the bases.

158. Perpendicular to the bases.

159. Parallel to one of the legs.

160. Drawn through a given point in one of the bases.

161. Drawn through a given point P.

162. Parallel to a given line L.

163. To divide a trapezoid by lines parallel to the bases into three parts proportional to the numbers 1, 2, 3.

164. To divide a trapezium into two equivalent parts by a line drawn through a given point in one of the sides.

165. To cut from a given polygon a triangle equivalent to a given square.

To divide a hexagon by a line drawn through a given point in one of the sides into:

166. Two equivalent parts.

167. Two parts in the ratio 2 : 3.

168. To inscribe in a given circle a rectangle of given area.

169. To inscribe in a given triangle a rectangle of given area.

170. To inscribe in a given parallelogram a rhombus of given area.

CHAPTER V.

REGULAR FIGURES.

§ 22. THEOREMS.

1. The side of a circumscribed equilateral triangle is equal to twice the side of the inscribed equilateral triangle. What is the ratio of their areas?

2. The area of a circumscribed square is equal to twice the area of the inscribed square. What is the ratio of their sides?

3. The apothem of an inscribed equilateral triangle is equal to one-half the side of the inscribed regular hexagon.

4. The apothem of an inscribed regular hexagon is equal to one-half the side of the inscribed equilateral triangle.

5. An inscribed regular hexagon is twice as large as the inscribed equilateral triangle.

6. A regular inscribed hexagon is one-half as large as the circumscribed equilateral triangle.

7. The area of a regular dodecagon is equal to three times the square of its radius.

8. Upon the six sides of a regular hexagon squares are constructed outwardly. Prove that the exterior vertices of these squares are the vertices of a regular dodecagon.

9. In a regular pentagon all the diagonals are drawn. Prove that another regular pentagon is thereby formed.

10. The apothem of an inscribed regular pentagon is equal to one-half the sum of the radius of the circle and the side of the inscribed regular decagon.

11. The side of an inscribed regular pentagon is equal to the hypotenuse of a right triangle of which the legs are the radius of the circle and the side of the inscribed regular decagon.

12. The radius of an inscribed regular polygon is the mean proportional between the apothem and the radius of the similar circumscribed regular polygon.

13. The area of a circular ring is equal to that of a circle whose diameter is a chord of the outer circle and a tangent to the inner circle.

14. The alternate vertices of a regular hexagon are joined by straight lines. Prove that another regular hexagon is formed. Find the ratio of the areas of the two hexagons.

15. If upon the legs of a right triangle semi-circumferences are described outwardly, the sum of the areas contained between these semi-circumferences and the semi-circumference passing through the three vertices, is equal to the area of the triangle.

16. If the diameter of a circle is divided into two parts, and upon these parts semi-circumferences are described on opposite sides of the diameter, these semi-circumferences will divide the circle into two parts which have the same ratio as the two segments of the diameter.

17. If in a circle two chords are drawn perpendicular to each other, and upon the four segments of these chords as diameters circles are described, the sum of the areas of the four circles will be equal to the area of the given circle.

§ 28. Numerical Exercises.

Note. When π occurs as a factor, take $\pi = 3\frac{1}{7}$ if English units are used, and $\pi = 3.1416$ if metric units are used.

In the following twenty-one exercises:

r = radius of circle.
ρ = apothem of inscribed regular polygon of n sides.
a = one side of inscribed regular polygon of n sides.
b = one side of inscribed regular polygon of $2n$ sides.
c = one side of circumscribed regular polygon of n sides.
F = area of inscribed regular polygon of n sides.
G = area of inscribed regular polygon of $2n$ sides.

In case approximate rational results are desired, $\sqrt{2} = 1.41421$, $\sqrt{3} = 1.73205$, $\sqrt{5} = 2.23606$.

	Given.	*Required.*
1.	r, a.	$\rho = \sqrt{r^2 - \frac{a^2}{4}} = \frac{1}{2}\sqrt{4r^2 - a^2}$.
2.	r, a.	$b = \sqrt{2r\left(r - \sqrt{r^2 - \frac{a^2}{4}}\right)}$.
3.	r, b.	$a = \frac{\sqrt{b^2(4r^2 - b^2)}}{r}$
4.	r, a.	$c = \frac{2ar}{\sqrt{4r^2 - a^2}}$.
5.	r, $n = 3$.	$a = r\sqrt{3}$, $\rho = \frac{r}{2}$.
6.	r, $n = 6$.	$a = r$, $\rho = \frac{r\sqrt{3}}{2}$

	Given.	*Required.*
7.	r, $n = 12$.	$a = r\sqrt{2-\sqrt{3}}$, $\quad \rho = \frac{r\sqrt{2+\sqrt{3}}}{2}$.
8.	r, $n = 4$.	$a = r\sqrt{2}$, $\quad \rho = \frac{r\sqrt{2}}{2}$.
9.	r, $n = 8$.	$a = r\sqrt{2-\sqrt{2}}$, $\quad \rho = \frac{r\sqrt{2+\sqrt{2}}}{2}$.
10.	r, $n = 10$.	$a = \frac{r}{2}(\sqrt{5}-1)$, $\quad \rho = \frac{r}{4}\sqrt{10+2\sqrt{5}}$.
11.	r, $n = 5$.	$a = \frac{r}{2}\sqrt{10-2\sqrt{5}}$, $\quad \rho = \frac{r}{4}(\sqrt{5}+1)$.
12.	a, n, ρ.	$F = \frac{na\rho}{2}$.
13.	a, n, r.	$F = \frac{na}{4}\sqrt{4r^2-a^2}$.
14.	a, n, r.	$G = \frac{nar}{2}$.
15.	a or r, $n = 3$.	$F = \frac{a^2\sqrt{3}}{4} = \frac{3r^2\sqrt{3}}{4}$.
16.	a or r, $n = 6$.	$F = \frac{3a^2\sqrt{3}}{2} = \frac{3r^2\sqrt{3}}{2}$.
17.	a or r, $n = 12$.	$F = 3r^2 = 3a^2(2+\sqrt{3})$.
18.	a or r, $n = 4$.	$F = a^2 = 2r^2$.
19.	a or r, $n = 8$.	$F = 2r^2\sqrt{2} = 2a^2(1+\sqrt{2})$.
20.	r, $n = 5$.	$F = \frac{5r^2}{8}\sqrt{10+2\sqrt{5}}$.
21.	r, $n = 10$.	$F = \frac{5r^2}{4}\sqrt{10-2\sqrt{5}}$.

Hints. In all cases, r, ρ, and $\frac{1}{2}a$ are the sides of a right triangle, r being the hypotenuse. —— In Ex. 9, a may be found either by a direct investigation or by making use of the result in Ex. 2. —— Ex. 10. Since the radius is divided in extreme and mean ratio, we have $r : a = a : r - a$. From this proportion find a in terms of r. —— In comparing the sides of the pentagon and the decagon, the expressions $\sqrt{5}+1$ and $\sqrt{5}-1$ occur. It should be noted with care that

$$\sqrt{5}+1 = \sqrt{5+2\sqrt{5}+1} = \sqrt{6+2\sqrt{5}},$$

and that $$\sqrt{5}-1 = \sqrt{5-2\sqrt{5}+1} = \sqrt{6-2\sqrt{5}}.$$

Ex. 14. The polygon of $2n$ sides is composed of $2n$ equal isosceles triangles. If in each triangle the radius of the circle is taken as the base, the altitude will be equal to $\frac{1}{2}a$. —— In Exs. 17, 19, 21, the result obtained in Ex. 14 may be immediately applied.

22. Apply the general formula obtained in Ex. 4 to the circumscribed equilateral triangle, square, and regular hexagon.

23. Find the area of a regular polygon of 24 sides inscribed in a circle whose radius is 1 foot.

24. Find the area of a regular dodecagon circumscribed about a circle whose radius is r.

25. Compare the areas of a circumscribed regular hexagon and an equilateral triangle inscribed in the same circle.

26. If the alternate vertices of a regular hexagon are joined by drawing six diagonals, prove that another regular hexagon is formed, and find the ratio of its area to that of the given hexagon.

27. Find the perimeter of a regular pentagon inscribed in a circle whose radius is 12 feet.

28. Find one side of a regular octagon circumscribed about a circle whose radius is r.

29. Given the side c of a circumscribed regular decagon, find the side of a regular decagon inscribed in the same circle.

30. Two regular octagons contain, respectively, 54 square feet and 62 square feet. Find, approximately, the side of a third regular octagon equal in area to the sum of the two given octagons.

31. The sides of three regular octagons are 3 feet, 4 feet, and 12 feet, respectively. Find the side of a regular octagon equal in area to the sum of the areas of the three given octagons.

32. The length of each side of a park in the shape of a regular decagon is 80 yards. Find the area of the park.

33. What will it cost, at \$1.80 per yard, to build a wall around a lot of land in the shape of a regular hexagon, containing 1039.24 square yards?

34. The perimeter of a regular hexagon is 480 feet, and that of a regular octagon is the same. Which is the greater in area, and by how much?

35. Find the different ways in which regular polygons may be employed for paving.

36. Find the circumference of a circle whose diameter is 7 inches.

37. Find the diameter of a circle whose circumference is 12 feet 10 inches.

38. The diameter of a carriage-wheel is $3\frac{1}{2}$ feet. How many revolutions does it make in traversing one-fourth of a mile?

39. If a carriage-wheel makes 220 revolutions in travelling half a mile, find its diameter.

40. What is the width of the ring between two concentric circumferences whose lengths are 440 feet and 330 feet?

41. In raising water from the bottom of a well, by means of a wheel, it is found that the wheel whose diameter is 2 feet 4 inches makes 30 revolutions in raising the bucket. Find the depth of the well.

42. Find the length of an arc of 36° in a circle whose diameter is 35 inches.

43. Find the angle subtended at the centre by an arc 6 feet 5 inches long, if the radius of the circle is 8 feet 2 inches.

44. The radius of a circle is 7 inches, and the length of an arc is the same. Find the angle subtended at the centre by the arc.

45. Find the angle subtended at the centre by an arc whose length is equal to the radius of the circle.

46. The radius of a circle is 5 feet 3 inches. What is the perimeter of a sector the angle of which is 45°?

47. If the angle subtended at the centre by an arc 5 feet 6 inches long is 72°, find the radius of the circle.

48. Find the length of an arc of 11° 15′ in a circle whose radius is 84 feet.

49. The radius of a circle is 14 feet. What is the length of the arc subtended by one side of the inscribed regular dodecagon?

50. Taking the length of a meridian of the earth as 40,000,000^{m}, find the length of an arc of 1″.

51. Find in kilometers the radius of the earth, assuming the circumference to be 40,000,000^{m}.

52. Two arcs have the same angular measure, but the length of one is twice that of the other. Compare the radii of the arcs.

53. Two arcs have the same length. Their angular measures are 20° and 30°, respectively. If the radius of the first arc is 6 feet, find the radius of the second arc.

54. Find the area of a circle whose diameter is 1 foot 2 inches.

55. Find the area of a circle whose circumference is 2 miles.

56. Find the diameter of a circle whose area is 6 acres.

57. Find the circumference of a circle whose area is 2 acres 176 square yards.

58. Find the area of a semicircle the radius of which is 14 feet.

59. The diameter of a circle is 56 feet. Find the side of a square equal in area to the circle.

60. The area of a square is 196 square feet. Find the area of the circle inscribed in the square.

61. A circular fish-pond which covers an area of 3 acres 880 square yards, is surrounded by a walk 3 yards wide. Find the cost of gravelling the walk, at 8 cents per square yard.

62. What must be the width of a walk around a circular plot of land containing 2 acres 2794 square yards, in order that the walk may contain exactly one-fourth of an acre?

63. The diameter of a circular grass-plot is 28 feet. Find the diameter of another circular plot which is just twice as large.

64. One side of a square piece of wood is 5 feet 10 inches long; out of it is cut the largest possible circle. How many square inches of wood are cut away?

65. Out of a circular piece of wood whose diameter is 3 feet 4 inches is cut the largest possible square. Find the length of its side.

66. Find the area between two concentric circles if their circumferences are 460 feet and 352 feet.

67. In a square room whose side measures 17 feet 6 inches is to be made a circular bath with its circumference touching the walls of the room. Find the area of the bath.

68. The perimeters of a circle, a square, and an equilateral triangle are each equal to 132 feet. Compare the areas of the three figures.

69. The radius of a circle is 3 feet. What is the radius of a circle 25 times as large?

70. Find the radius of a circle equal in area to the difference between the areas of two given circles with the radii r, r'.

71. What is the ratio of the area of a circle to that of the inscribed equilateral triangle?

72. What is the ratio of the area of the circle inscribed in an equilateral triangle to that of the circle circumscribed about the same triangle?

73. The sum of the areas of a circle and the inscribed equilateral triangle is 3^{qm}. Find the area of each figure.

74. What will it cost to pave a circular court 30 feet in diameter, at 54 cents per square foot, leaving in the centre a hexagonal space, each side of which measures 2 feet?

75. A circle 18 feet in diameter is divided into three equivalent parts by two concentric circumferences. Find the radii of these circumferences.

76. Calculate the expense of making a moat round a circular island, at 60 cents per square yard, the diameter of the island being 525 feet, and the breadth of the moat being 21 feet 6 inches.

77. The chord of half an arc is 17 feet, and the height of the arc, 7 feet. Find the diameter of the circle.

78. The chord of half an arc is 12 feet, and the radius of the circle is 18 feet. Find the height of the arc.

79. The height of an arc is 9 inches, and the diameter of the circle is 125 feet. Find the chord of half the arc.

80. The chord of an arc is 24 inches, and the height of the arc is 9 inches. Find the diameter of the circle.

81. The span (chord) of a bridge in the form of a circular arc is 96 feet, and its height above the stone piers is 12 feet. Find the radius of the circle.

82. The height of the arch of a bridge, the form of which is the arc of a circle, is 24 feet, and the radius of the circle is 312 feet. Find the span of the arch.

83. The radius of a circle is 12 feet. The chords which subtend two arcs AB, BC are 6 feet and 9 feet, respectively. Find the chord AC.

84. The lengths of two chords drawn from the same point in the circumference of a circle to the extremities of a diameter are 6 feet and 8 feet. Find the area of the circle.

85. The chord of an arc is 32 inches, and the radius of the circle is 34 inches. Find the length of the arc.

86. The diameter of a circle is 106 feet. Find the lengths of the two arcs into which a chord 90 feet long will divide the circumference.

87. The span of a bridge, the form of which is the arc of a circle, is 200 feet, and the height of the crown above the stone piers is 42 feet. Find the length of the arch.

88. Find the area of a sector if the radius of the circle is 56 feet, and the angle at the centre is $22\frac{1}{2}°$.

89. The area of a sector is 385 square feet, and the angle of the sector is 36°. Find the radius of the circle and the perimeter of the sector.

90. The chord of an arc is 40 feet, and the radius of the circle is 21 feet. Find the area of the sector.

91. Find the area of a segment if the chord of the arc is 56 feet, and the radius of the circle is 35 feet.

92. The chord of a segment is 40 feet, and its height is 9 feet. Find its area.

93. A room 20 feet long and 15 feet wide has a recess at one end in the shape of the segment of a circle, the chord of the recess being 15 feet, and its greatest width 4 feet. Find the area of the whole room.

94. Find how many square feet of brick are required in blocking up one of the arches of a railway viaduct, if the span of the arch is 60 feet, height above the piers is 20 feet, distance from the ground to the spring of the arch is 20 feet.

Compute in terms of the radius r of the circle the area of the segment subtended by one side of the inscribed:

95. Equilateral triangle.

96. Regular hexagon.

97. Square.

98. Regular octagon.

99. Regular pentagon.

100. Regular decagon.

101. Find the area of a circle in which a chord 3 feet long subtends an arc of 120°.

102. Find the area of a circle if the area of the inscribed regular hexagon is 10 square feet.

103. Find the area of a segment whose arc is 300°, the radius of the circle being 1 foot.

104. Find the ratio of the two segments into which a circle is divided by a perpendicular erected at the middle point of one of the radii.

105. The radius of a circle is 7^m. Find the areas of the two segments into which the circle is divided by a chord equal to the radius.

106. Two tangents drawn from a point exterior to a circle whose radius is 1.3^m intercept upon the circumference an arc whose length is 0.57^m. Required the angle formed by the two tangents.

107. The areas of two concentric circles are as 5 to 8. The area of that part of the ring which is contained between two radii making the angle 45° is 300 square feet. Find the radii of the two circles.

108. What is the altitude of a rectangle equivalent to a sector whose radius is 15 feet, if the base of the rectangle is equal to the arc of the sector?

109. The radii of two concentric circles are 3 feet and 5 feet. What is the altitude of a trapezoid equivalent to the ring and having for bases the lengths of the two circumferences?

110. Find the radius of a circle if its area is doubled by increasing the radius 1 foot.

111. The sides of a rectangle inscribed in a circle are a, b. Find the area of the circle in terms of a and b.

112. The radius of a circle is r. Upon the chord of a quadrant as diameter a semi-circumference is described outwardly. Find the area of the surface contained between the semi-circumference and the arc of the quadrant.

113. The radius of a circle is r. Through a point exterior to the circle two tangents are drawn making an angle of 60°. Find the area of the figure bounded by the tangents and the intercepted arc.

114. Three equal circles are described, each touching the two others. If the common radius is r, find the area contained between the circles.

115. In a quadrant whose radius is r a circle is inscribed. Find (i.) the area of this circle, (ii.) the area comprised between the circle and the arc of the quadrant.

116. Upon each side of a square as a diameter semi-circumferences are described within the square forming four leaves (*Fig.* 93). If a side of the square is a, find the sum of the areas of the four leaves.

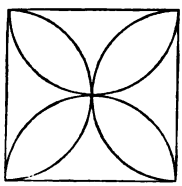


Fig. 93.

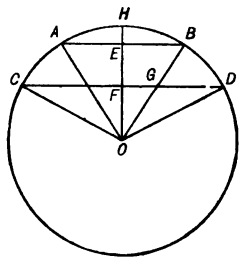


Fig. 94.

117. In a circle whose radius is r, two parallel chords are drawn on the same side of the centre, one, AB, equal to the side of the inscribed regular hexagon, the other, CD, equal to the side of the inscribed equilateral triangle. Find the area of the portion of the circle contained between the two chords.

This question may be solved algebraically by computing the difference between the segments CHD and AHB (*Fig.* 94). But the following geometrical solution is shorter. Let E, F, H be the middle points of AB, CD, and arc AB, respectively, and let G be the intersection of CD and OB. The rt. $\triangle EOB$ and FOD are equal (why?). By subtracting the common part FOG and adding GBD it becomes evident that the figures $FEBD$ and BOD are equivalent. But sector BOD is equivalent to sector HOB, because the $\triangle BOD$ and HOB are each equal to 30°. Therefore, area $ABCD$, which is double that of $FEBD$, is equivalent to sector AOB, which is double the sector HOB. Now sector $AOB = \frac{1}{6}\pi r^2$, $\therefore$ area $ABCD = \frac{1}{6}\pi r^2$.

118. Given p, P the perimeters of regular polygons of n sides inscribed in and circumscribed about a given circle, find p', P' the perimeters of regular polygons of $2n$ sides inscribed in and circumscribed about the same circle.

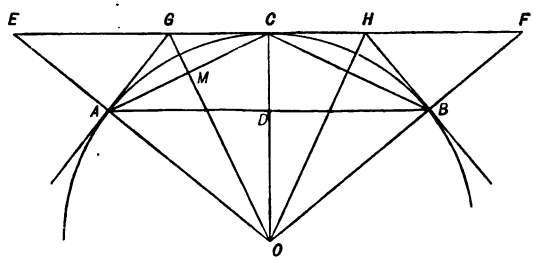


Fig. 95.

Solution. Let (*Fig.* 95) AB be one side of the polygon whose perimeter is p, C the middle point of the arc AB. Join AC, draw through C a tangent meeting OA and OB produced in E and F respectively; also draw tangents through A and B, meeting EF in G and H respectively; join OG, and let OG cut AC in M.

Then AC = side of polygon whose perimeter is p';
EF = side of polygon whose perimeter is P;
GH = side of polygon whose perimeter is P.

Now, $$p : P = OA : OE = OC : OE;$$

$$OC : OE = CG : GE.$$

$$\therefore p : P = CG : GE.$$

By the theory of proportions,

$$p : P + p = CG : CE.$$

But, $$CG = \frac{P'}{4n},\ CE = \frac{P}{2n}.$$

Substituting these values of CG and CE, and solving for P', we obtain

$$P' = \frac{2Pp}{P+p}.$$

Secondly, the $\triangle ADC$ and CGM are similar.

$$\therefore AC : AD = CG : CM.$$

Now, $$AC = \frac{p'}{2n},\ AD = \frac{p}{2n};\ CG = \frac{P'}{4n};\ CM = \frac{p'}{4n}.$$

Substituting these values, and solving for p', we obtain

$$p' = \sqrt{P'p}.$$

119. Show how to compute approximately the value of π by means of the formulas deduced in the last exercise.

If C denote the circumference of a circle, and D its diameter, then by definition, $\pi = \frac{C}{D} = \frac{C}{2r}$. If we make $r = 1$, then $\pi = \frac{C}{2}$. Now, by means of the formulas of the last exercise, we can obtain the value of C to as close a degree of approximation as we please. For example, beginning with the square, we have $p = 4\sqrt{2}$, $P = 8$; whence we find the perimeters p', P' of the inscribed and circumscribed regular octagons, and then those of regular polygons of 16 sides, etc. For polygons of 256 sides, $p' = 6.28303$, $P' = 6.28350$. Since the value of C lies between those of p' and P', therefore the value $C = 6.283$ is correct to three decimal places; and the same is true of the value $\pi = \frac{C}{2} = 3.141$.

This method of determining the value of π is the one commonly given in text-books, and is known as the Method of Perimeters.

120. Given the radius r and the apothem ρ of an inscribed regular polygon of n sides: find the radius r' and the apothem ρ' of an isoperimetrical regular polygon of $2n$ sides.

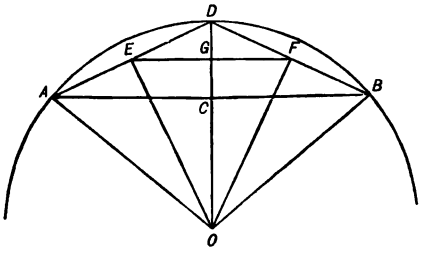


Fig. 96.

Hints. Let AB (*Fig.* 96) be one side of the given polygon, C its middle point. Produce OC to D, join AD and BD, and join E, F, the middle points of AD, BD, by a line EF, cutting OD in G. EF = side of required polygon (why?); $\angle EOF$ = angle at its centre (why?). Hence, $OE = r'$, $OG = \rho'$. The following results are now readily found:

$$\rho' = \frac{r+\rho}{2}, \qquad r' = \sqrt{\frac{r(r+\rho)}{2}}.$$

It is evident, both from the figure and from these values of ρ' and r', that $\rho' > \rho$, and that $r' < r$.

121. Show how to compute approximately the value of π by the aid of the formulas deduced in the last exercise.

For a side whose square is 1, we have perimeter = 4, $r = \frac{1}{2}\sqrt{2}$, $\rho = \frac{1}{2}$.

Now, with the aid of the formulas of the last exercise, we can compute the radius and the apothem of the isoperimetrical regular octagon, then those of isoperimetrical regular polygons of 16, 32, 64, etc., sides. As we increase the number of sides the radius decreases, the apothem increases, while the perimeter remains constantly equal to 4. The radius and the apothem both approach, as a common limit, the radius of a circle whose circumference is equal to 4; in other words, the value of the radius of this circle lies between the values of the radius and of the apothem of the polygon, and hence can be found to any desired degree of approximation.

For a polygon of 8192 sides, $\rho' = 0.6366196.....$, $r' = 0.6366196.....$: hence, the value 0.6366196 must be the correct value to seven places of decimals of the radius of a circle whose circumference is equal to 4. Whence, we have

$$\pi = \frac{C}{2r} = \frac{4}{2 \times 0.6366196} = 3.1415926.$$

This method of computing the value of π is known as the Method of Isoperimeters.

§ 24. Problems.

1. Construction of a Regular Polygon. The parts of a regular polygon are the number of sides n, the length of one side a, the radius r, and the apothem ρ. The angle at the centre and the angle of the polygon are determined by the value of n, and therefore are not independent parts of the polygon. Neither can any two of the three parts a, r, ρ, have arbitrary values, but only those values which will make the angle at the centre an aliquot part of 360°.

A regular polygon is determined by *two* independent parts, because it is divisible into equal right triangles. The sides of each triangle are $r, \rho, \frac{1}{2}a$; the acute angles are the angle of the polygon and half of the angle at the centre, both of which are known if n is given.

From what has been said, it follows that the construction of a regular polygon falls under one of three cases:

(i.) To construct a regular polygon, given r and n.

(ii.) To construct a regular polygon, given ρ and n.

(iii.) To construct a regular polygon, given a and n.

In each case, the values of n for which an *exact* construction can be given are limited to the terms of the four series in No. 231. Moreover, large values of n are excluded on account of the difficulty in effecting the construction.

2. To find the simplest construction for the sides of an inscribed regular pentagon and an inscribed regular decagon.

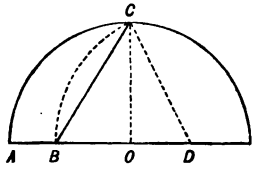


Fig. 97.

Let BO (*Fig.* 97) be one side of the inscribed regular decagon.

Then (No. 228) $\overline{OB}^2 = r(r - OB)$,

whence, by solving, $OB + \frac{r}{2} = \sqrt{r^2 + \frac{r^2}{4}}$

Therefore, $OB + \frac{r}{2}$ is equal to the hypotenuse of a right triangle of which the legs are r and $\frac{r}{2}$.

Hence, to construct OB, draw $OC \perp$ to AO, take $OD = \frac{r}{2}$; from D as centre, with DC as radius, cut OA in B: then OB will be one side of the inscribed regular decagon. Also,

$$\begin{aligned}
\overline{BC}^2 &= r^2 + \overline{OB}^2 \\
&= r^2 + \frac{r^2}{4}(\sqrt{5} - 1)^2 \qquad (\S\ 23, \text{Ex. } 10.) \\
&= r^2 + \frac{r^2}{4}(6 - 2\sqrt{5}) \\
&= \frac{4r^2 + 6r^2 - 2r^2\sqrt{5}}{4} \\
&= \frac{r^2}{4}(10 - 2\sqrt{5}),
\end{aligned}$$

$$\therefore BC = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}.$$

Therefore (§ 23, Ex. 11), BC is equal to the side of the inscribed regular pentagon.

To inscribe in a given circle:

3. An equilateral triangle.
4. A square.
5. A regular hexagon.
6. A regular octagon.
7. A regular pentagon.
8. A regular decagon.

To circumscribe about a given circle:

9. An equilateral triangle.
10. A square.
11. A regular hexagon.
12. A regular octagon.
13. A regular pentagon.
14. A regular decagon.

To construct upon a given line as one side:

15. An equilateral triangle.
16. A square.
17. A regular hexagon.
18. A regular octagon.
19. A regular pentagon.
20. A regular decagon.

21. To construct upon a given line as one side a regular dodecagon.

22. To construct upon a given line as one side a regular polygon of 15 sides.

23. To construct an angle of 36°.

24. To construct an angle of 9°.

25. To construct an angle of 12°.

26. In a given circle, to construct a mean proportional between a chord of 30° and a chord of 150°.

27. To transform a given regular octagon into a square.

28. To construct a regular hexagon, given one of the shorter diagonals.

29. To construct a regular pentagon, given one of the diagonals.

30. To inscribe in a given circle a polygon of n sides, n being any whole number.

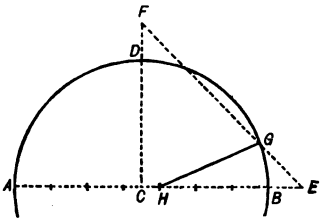


Fig. 98.

The following construction is found in most cases to be sufficiently exact for practical purposes.

Divide the diameter AB (*Fig.* 98) into n equal parts (in the Figure $n = 7$). Draw the radius $CD \perp$ to AB, produce CB to E, and CD to F, making BE and DF each equal to one of the parts of the diameter; join EF, cutting the circle for the first time in G. Then the line GH joining G and the *third* point of division of AB, counting from B, will be very nearly equal to one side of the inscribed polygon of n sides.

For $n = 3$ and $n = 4$, this construction is impossible; for $n = 5$ it is useless, on account of its inaccuracy; but for $n > 5$ it gives a very close approximation to the exact value of the side required.

31. To construct a regular decagon equivalent to a given regular pentagon.

32. To construct a circle in which the inscribed regular octagon shall have a perimeter equal to that of a given square.

33. To draw through a given point a line which shall divide a given circumference into two parts in the ratio of 3 to 7.

34. To construct a circumference equal to the sum of two given circumferences.

35. Three given circumferences are denoted by c, c', c''; to construct a circumference equal to $\frac{1}{2}c + \frac{3}{4}c' - \frac{5}{6}c''$.

36. To construct a circle equivalent to the sum of two given circles.

37. To construct a circle equal in area to three times a given circle.

38. To construct a circle equal in area to three-fourths of a given circle.

39. To divide a given circle by a concentric circumference into two equivalent parts.

40. To divide a given circle by concentric circumferences into four equivalent parts.

41. To inscribe four equal circles in a given regular octagon so that each circle shall touch two other circles and one side of the octagon.

42. To inscribe in a given circle five equal circles so that each circle shall touch two other circles and the circumference of the given circle.

CHAPTER VI.

THE ALGEBRAIC METHOD.

§ 25. Construction of Algebraic Expressions.

The lengths of lines, expressed in terms of a common unit, are called **Linear Factors.** Abstract numbers and symbols representing abstract numbers are called **Coefficients.**

In the following exercises the first letters of the alphabet, a, b, c, etc., are employed as linear factors, and the letters m and n denote coefficients.

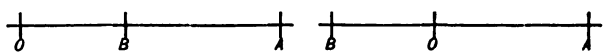
I. *Elementary Expressions.*

1. To construct $x = a + b$.

In other words, to construct a line equal to the sum of two given lines. The construction is obvious.

2. To construct $x = a - b$.

Draw $OA = a$, then from A take in the opposite direction $AB = b$; OB represents the value of x.



If $b > a$, the value of x is negative. In this case, the problem has no meaning if absolute magnitude alone is considered. But if *position* is also taken into account, the negative value of x is represented by a line OB extending from O *in the opposite direction* to that of a positive value of x, or that of the positive line OA.

This exercise furnishes a simple illustration of the principle of Descartes, that *contrary signs in Algebra correspond to opposite directions in Geometry.* Or, stated more exactly, if a positive value of a

linear expression is properly represented by a line drawn from a certain point in a certain direction, a negative value will be represented by a line drawn from the same point in the opposite direction.

3. To construct $x = a + b - e + d - e + f$.

4. To construct $x = na$.

5. To construct $x = \frac{a}{n}$.

6. To construct $x = \frac{ma}{n}$.

The solution consists in dividing a in the ratio $m : n$.

7. To construct $x = ab$.

In this case, x is represented by the area of a rectangle whose sides are equal to a and b.

8. To construct $x = a^2$.

9. To construct $x = \frac{ab}{c}$.

x is the fourth proportional to a, b, and c, and constructed by means of No. 139 or 162 or 163.

10. To construct $x = \frac{a^2}{b}$.

x is the third proportional to a and b, and is constructed by means of No. 157 or 158 or 164.

11. To construct $x = \sqrt{ab}$.

x is the mean proportional between a and b, and is constructed by means of No. 157 or 158 or 164.

12. To construct $x = \sqrt{a^2 + b^2}$.

x is equal to the hypotenuse of a right triangle whose legs are a and b.

13. To construct $x = \sqrt{a^2 - b^2}$.

x is equal to one leg of a right triangle in which the hypotenuse is a, and the other leg b. Also, since $\sqrt{a^2 - b^2} = \sqrt{(a+b)(a-b)}$, x is the mean proportional between $a + b$ and $a - b$.

14. To construct $x = a\sqrt{2}$.

x is equal to the diagonal of a square whose side is a.

15. To construct $x = \frac{a}{2}\sqrt{3}$.

x is equal to the altitude of an equilateral triangle whose side is a.

16. To construct $x = a\sqrt{5}$.

x is equal to the hypotenuse of a right triangle whose legs are a and $2a$.

17. To construct $x = a\sqrt{m}$.

x is the mean proportional between a and ma.

18. To construct $x = \frac{ab}{\sqrt{a^2 + b^2}}$.

x is equal to the altitude of a right triangle whose legs are a and b.

19. To construct $x = \sqrt{a^2 + b^2 - ab}$.

x is the third side of a triangle in which the other sides are a and b, and the angle included by them is 60°.

20. To construct $x = \sqrt{a^2 + b^2 + ab}$.

x is the third side of a triangle in which the other sides are a and b, and the angle included by them is 120°.

II. *Expressions Reducible to Elementary Expressions.*

To construct the following expressions:

21. $x = \sqrt{a^2 + b^2 + c^2 + d^2}$. **23.** $x = a\sqrt{3}$.

22. $x = \sqrt{a^2 - b^2 + c^2 - d^2}$. **24.** $x = a\sqrt{19}$.

Three different constructions are suggested by putting $a\sqrt{19}$ into the forms $\sqrt{a \times 19a}$,

$\sqrt{(4a)^2 + (2a)^2 - a^2}$,

and $\sqrt{(5a)^2 - (2a)^2 - a^2 - a^2}$.

25. $x = a\sqrt{15}$. **26.** $x = a\sqrt{21}$.

27. $x = \frac{2abc}{de} = \frac{2ab}{d} \times \frac{c}{e}$. **28.** $x = \frac{abcd}{efg}$.

29. $x = \dfrac{3a^2b}{c^2} = 3\dfrac{\dfrac{ab}{c} \times a}{c} = 3\dfrac{\dfrac{a^2}{c} \times b}{c}$.

30. $x = \dfrac{2a^3b^2c}{3d^2e^2f}$.

31. $x = \dfrac{ab}{a+b}$.

32. $x = \sqrt{2ab}$.

33. $x = \sqrt{\dfrac{abc}{d}}$.

34. $x = \sqrt{\dfrac{3a^2}{2}}$.

35. $x = \sqrt{ab + cd}$.

36. $x = \sqrt{2a^2 - ab}$.

37. $x = \dfrac{a^2 - b^2}{2c} = \dfrac{1}{2} \times \dfrac{(a+b)(a-b)}{c}$.

38. $x = \dfrac{a}{2}(\sqrt{5} - 1) = \dfrac{a}{2}\sqrt{5} - \dfrac{a}{2}$.

39. $x = \dfrac{a}{2}(1 + \sqrt{3}) = \dfrac{a}{2} + \sqrt{a^2 - \left(\dfrac{a}{2}\right)^2}$.

40. $x = a\sqrt{2 - \sqrt{2}} = \sqrt{2a^2 - a\sqrt{2a^2}}$.

41. $x = a\sqrt{1 + \sqrt{2} + \sqrt{3}} = \sqrt{a(a + a\sqrt{2} + a\sqrt{3})}$.

42. $x = \sqrt{\dfrac{ab}{\sqrt{5}-1}} = \sqrt{\dfrac{ab(\sqrt{5}+1)}{4}} = \frac{1}{2}\sqrt{a(b + b\sqrt{5})}$.

43. $x = \sqrt[4]{abcd} = \sqrt{\sqrt{ab} \times \sqrt{cd}}$.

44. $x = \sqrt[4]{a^4 + a^2b^2} = \sqrt{a\sqrt{a^2 + b^2}}$.

45. $x = \dfrac{a^3 + b^3}{a^2 + b^2} = \dfrac{(a+b)(a^2 + b^2 - ab)}{a^2 + b^2}$.

46. To construct the roots of the equation $x^2 - ax + b^2 = 0$.

It is shown in Algebra that if a complete equation of the second degree is written in the form $x^2 + px + q = 0$, the algebraic sum of the roots $= -p$, the product of the roots $= q^2$.

Therefore, in the present case, if x_1 x_2 denote the roots,

$$x_1 + x_2 = a,$$
$$x_1 \times x_2 = b^2.$$

These relations show that the roots, if real, are both positive, since their sum a and product b^2 are both positive; and the question is reduced to the problem: To construct two lines, given their sum a and their product b^2.

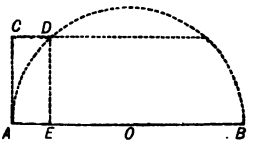


Fig. 99.

Draw (*Fig.* 99) $AB = a$, describe upon AB as diameter a semicircle, then draw $AC \perp$ to AB and equal to b, $CD \parallel$ to AB and meeting the circle in D, $DE \perp$ to AB, meeting AB in E. The required lines are AE and EB. (No. 157).

If $b > \frac{a}{2}$, or $a < 2b$, CD will not meet the circle, and the roots are imaginary. If $a = 2b$, CD touches the circle, and the roots are equal each to $\frac{a}{2}$.

47. To construct the roots of the equation $x^2 + ax + b^2 = 0$.

Since this equation is obtained from the equation $x^2 - ax + b^2 = 0$ by changing x to $-x$, its roots will be the same as those of $x^2 - ax + b^2 = 0$, with contrary signs. Therefore, their absolute values may be constructed exactly as in the last exercise.

48. To construct the roots of the equation $x^2 - ax - b^2 = 0$.

In this case the roots have contrary signs, since their product $-b^2$ is negative. Let x_1 denote the positive root, x_2 the *absolute value* of the negative root; then

$$x_1 - x_2 = a,$$
$$x_1 \times x_2 = b^2,$$

and the question is reduced to the problem: To construct two lines, given their difference a and their product b^2.

With O (*Fig.* 100) as centre, and $\frac{a}{2}$ as radius, describe a circle. Through any point A of the circumference draw a tangent $AB = b$; then draw BO meeting the circumference in C and D. The required lines are BC and BD. (No. 164.)

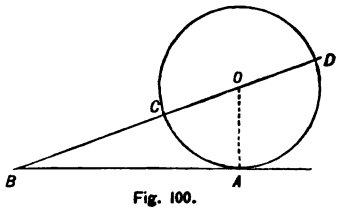


Fig. 100.

49. To construct the roots of the equation $x^2 + ax - b^2 = 0$.

This equation is obtained from the equation $x^2 - ax - b^2 = 0$ by changing x to $-x$; therefore its roots are the same as those of $x^2 - ax - b^2 = 0$, with contrary signs, and their absolute values are constructed in the same way.

NOTE. Every complete quadratic equation containing linear factors can be put into one of the four forms discussed in Exs. 46–49.

§ 26. HOMOGENEITY OF ALGEBRAIC EXPRESSIONS.

The **Degree** of a rational algebraic term, containing linear factors, is equal to the number of linear factors in the numerator diminished by the number of linear factors in the denominator.

If the number of linear factors in numerator and denominator is the same, as in $\frac{a}{b}$, $\frac{2a^2b}{3cd^2}$, the term is said to be of the zero degree. Such terms are equivalent to abstract numbers or coefficients.

Terms of the first degree always represent lengths; for they are always reducible to the product of a coefficient and a linear factor. (Ex. 4.)

Examples: $6a$, $\frac{ab}{c} = \frac{a}{c} \times b$, $\frac{3a^3b^3c}{4d^3e^3f} = \frac{3a^3b^3}{4d^3e^3f} \times c$.

Terms of the second degree always represent areas; for they are always reducible to the product of a coefficient and two linear factors. (Ex. 7.)

Examples: nab, $\frac{6a^2}{b} = \frac{6a}{b} \times a^2$, $\frac{8a^4b^3}{c^3d^2} = \frac{8a^4b}{c^3d^2} \times b^2$.

Terms of the third degree always represent volumes, and do not occur in Plane Geometry.

Terms of degrees higher than the third have no geometrical signification.

The degree of an irrational term (*i.e.* a term under a radical sign), is found by dividing the degree, estimated without regard to the radical sign, by the index of the root. Irrational terms, like rational terms, if of the first degree, represent lengths; if of the second degree, areas; and if of the third degree, volumes.

Examples: $\sqrt{\frac{abc}{d}}$ is of the first degree;

$\sqrt[4]{\frac{2a^3b^4}{c^3d}}$ is also of the first degree;

$\sqrt[3]{\frac{na^5b^3}{cd}}$ is of the second degree.

An algebraic expression consisting of two or more terms is said to be **homogenous** if all the terms are of the same degree.

Examples:

$5a + \frac{mab}{c} - \frac{na^2}{d}$ is a homogeneous expression of the first degree;

$\sqrt{a^2 + b^2 + c^2 + d^2}$ is also of the first degree;

$3a^2b - 4ab^2 + 6b^3$ is of the third degree.

The algebraic expressions of the values of geometric magnitudes are always homogeneous; for all the parts which

compose the whole must be of the same kind as the whole (either lines or surfaces or volumes), and therefore, the terms which represent the values of the parts must be of the same degree.

Homogeneous expressions of the first degree represent lengths; of the second degree, areas; of the third degree, volumes. Homogeneous expressions of degrees higher than the third have no geometrical meaning.

From the degree of an expression, therefore, it can be inferred whether it is capable of geometric construction, and, if so, what kind of geometric magnitude it represents.

The principle of homogeneity also affords one test of the correctness of expressions. If, for example, we obtain an expression for a certain line which is either not homogeneous or not of the first degree, we may conclude at once that our investigation is somewhere at fault.

§ 27. Examples of the Algebraic Method.

I. To divide a given line a into two parts such that the difference of their squares shall be equal to their product.

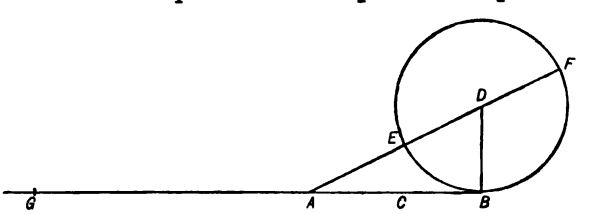


Fig. 101.

Analysis (*Fig.* 101). Let $AB = a$, and let C be the required point of division; then the problem is solved when AC has been determined. Let $AC = x$; then $BC = a - x$, and

$$x^2 - (a - x)^2 = x(a - x);$$

whence,

$$x^2 + ax - a^2 = 0,$$

an equation of the form considered in Ex. 49. Its roots are

$$x_1 = -\frac{a}{2} + \sqrt{\frac{a^2}{4} + a^2} = \sqrt{\left(\frac{a}{2}\right)^2 + a^2} - \frac{a}{2};$$

$$x_2 = -\frac{a}{2} - \sqrt{\frac{a^2}{4} + a^2} = -\left[\sqrt{\left(\frac{a}{2}\right)^2 + a^2} + \frac{a}{2}\right].$$

Construction. We may proceed as in Ex. 48, or as follows: Draw $BD \perp AB$ and equal to $\frac{1}{2}a$. Join AD. With D as centre and DB as radius, describe a circle cutting AD in E and AD produced in F. Then, *in absolute value*, $x_1 = AE$, $x_2 = AF$. In sign x_1 is positive, x_2 negative; therefore, upon AB take $AC = AE$, and upon BA produced to the left, take $AG = AF$.

Discussion. The point C falls between A and B (Why?) and is evidently the only solution of the problem in the strict sense of the enunciation. If, however, the problem had been stated as follows: *To produce a given line BA to a point G so that $GB^2 - GA^2 = GB \times GA$*; then, putting $AG = x$, the equation would have been

$$(a + x)^2 - x^2 = x(a + x),$$

the positive root of which,

$$x = \frac{a}{2} + \sqrt{a^2 + \frac{a^2}{4}},$$

is identical in absolute value with the negative root x_2 of the equation of the original problem. It appears, then, that the algebraic solution of the given problem supplies also the solution of a closely-related problem, when both roots of the equation are considered. And this suggests a mode of stating the given problem in a generalized form, so as to include both problems:

Upon a line passing through two given points A, B, to find a point C such that the difference between the squares of its distances from A and B shall be equal to their product.

It may be noticed that the construction for the point C (*Fig.* 101) is the same as that required in the problem, *To divide a given line in extreme and mean ratio;* in fact, the proportion to be satisfied in extreme and mean ratio, $a : x = x : a - x$ gives the equation

$$x^2 + ax - a^2 = 0.$$

II. In a given square, to inscribe an equilateral triangle, having one vertex in common with the square, and the other two vertices in the sides of the square.

Analysis. (*Fig.* 102.) Let $ABCD$ be the given square, AEF the triangle required. If $AB = a$, $BE = x$, then $\overline{AE}^2 = a^2 + x^2$, and $\overline{EF}^2 = 2(a - x)^2$. But $AE = EF$; therefore, $a^2 + x^2 = 2(a - x)^2$. Solving this equation for x, we have $x = 2a \pm a\sqrt{3}$.

Both values of x are positive, but the greater value is greater than a, and must be rejected if the sides of the square are regarded as lines limited by the vertices.

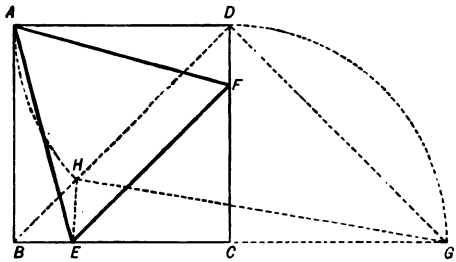


Fig. 102.

Construction. Produce BC to G, making $CG = a$. Join DG and DB, upon DB take $DH = DA$, and join GH; then $GH = a\sqrt{3}$. Upon GB take $GE = GH$; then $BE = 2a - a\sqrt{3}$, and E is a vertex of the triangle required. From A as a centre, with AE as radius, cut CD in F. Join EF. AEF is the triangle required.

A *proof* that AEF is an equilateral triangle may readily be given, and is left as an exercise for the learner.

Discussion. The construction of the other root $2a + a\sqrt{3}$ determines a point K in BC produced. If L denote a point in DC produced such that $AL = AK$, it is easy to prove that the $\triangle AKL$ will also be equilateral. The $\triangle AKL$, therefore, is also a solution of the problem, provided the sides of the square are regarded as lines unlimited in length.

III. Having given a circle and a point P, to draw through P a secant such that the exterior segment shall be equal to the interior segment.

Analysis. Let O be the centre of the given circle, r its radius, $OP = a$; then, if x denote the exterior segment, we have (No. 163),

$$2x^2 = a^2 - r^2;$$

whence,

$$x = \pm\sqrt{\frac{a^2 - r^2}{2}}.$$

The *Construction* and *Proof* present no difficulties requiring explanation.

Discussion. The negative value of x corresponds to no closely related problem, and must be rejected. The problem from its nature is impossible either if $a < r$, or, if $x > 2r$; that is, if $a > 3r$; in the first case, the impossibility is indicated by the value of x becoming imaginary.

IV. From two given lines a, b, to take away equal parts in such a way that the sum of the remainders shall be equal to a given line c.

Analysis. Let x denote one of the equal parts; then, by the conditions,

$$(a - x) + (b - x) = c;$$

whence,

$$x = \frac{a + b - c}{2}.$$

The *Construction* is obvious.

Discussion. If c is greater than $a + b$, the value of x is negative. In order to see how the statement of the problem should be modified to cover this case, substitute $-x$ for x in the original equation; it now takes the form $(a + x) + (b + x) = c$, which is the direct algebraic expression of the problem: *to add equal parts to two given lines so that their sum shall be equal to a third line.*

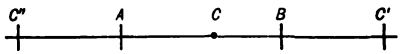


Fig. 103.

V. The intensities at unit distances of two luminous points A, B are represented by a, b, respectively; to find a point C in the line AB, which is equally illuminated by the two points of light.

Analysis. (*Fig.* 103.) Let $AB = d$, $AC = x$; then, since by a law of optics the quantities of light received at C from A and B are inversely as the squares of the distances, we have,

$$\frac{a}{x^2} = \frac{b}{(d - x^2)}. \qquad (1)$$

This equation, if we put $\frac{b}{a} = m^2$, becomes

$$(d-x)^2 = m^2x^2;$$

whence, $$d - x = \pm mx.$$

If x_1, x_2 denote the true values of x, we have,

$$x_1 = \frac{d}{1+m};$$

$$x_2 = \frac{d}{1-m};$$

and it appears that in general there are two points equally illuminated in the line AB.

The *Construction* presents no difficulty.

Discussion. The expression for x_2 gives rise to three cases, according as m is less than, equal to, or greater than, 1.

CASE I. $m < 1$. Then x_1 and x_2 are both positive, x_1 is $< d$, but x_2 is $> d$. Therefore, there is one point C between A and B, and another point C' to the right of B. This is otherwise evident; for, if m is < 1, then b is $< a$, that is, B is the feebler light, and the two points of equal illumination must both be nearer to B. In this case, $BC = x - d$, but the squares of $x - d$ and $d - x$ being equal, equation (1) still holds true.

CASE II. $m = 1$. Then $x_1 = \frac{d}{2}$, $x_2 = \infty$. This result means that the point C is now half way between A and B, and that the point C' no longer exists.

CASE III. $m > 1$. In this case x_1 is positive and less than d, while x_2 is negative. The points equally illuminated, therefore, are a point C between A and B, and a point C'' *to the left* of A. This result is otherwise an evident consequence of the fact that now A is the feebler light. If in equation (1) we write $-x$ for x, the equation becomes

$$\frac{a}{x^2} = \frac{b}{(d+x)^2}.$$

But this form corresponds exactly to the position of C'', since, in this case, $C''B = d + x$.

Thus, equation (1) covers all possible cases in the solution of the problem, provided that we regard x as positive or negative, according as the point C is situated to the right or to the left of A.

The foregoing examples show how Algebra may be applied in the solution of problems, and what the advantages of the algebraic method are. By an algebraic analysis we obtain for the value of an unknown length a general expression which includes all possible hypotheses respecting the values of the given lengths; so that a mere examination of this expression, without a figure, usually suffices to determine the limits of possibility of the problem. By introducing the conception of negative lines the algebraic method enables us to group together related problems, to see clearly their exact relations, and to bring them, when possible, under a single generalized form of statement.

In applying this method, the following principles must be observed:

I. As many equations are necessary as there are unknown lengths (usually only one) to be determined.

II. An imaginary value, obtained for an unknown length x indicates that the problem is impossible.

III. A negative value of x must be rejected if x denote simply an absolute length without reference to position or direction (*e.g.* the radius of a circle).

IV. A negative value of x, when the position or direction of x enters into account, usually corresponds to a modified or extended statement of the problem; to see what this should be, substitute $-x$ for x in the original equation and then translate the new equation into the statement of a problem.

V. If a positive value of a length x is properly represented by a line drawn from a given point in a certain direction, then a negative value of x, if admissible from the nature of the question, will be represented by a line drawn from the same point in the *opposite* direction. (Principle of Descartes.)

§ 28. Problems.

Group I. *Classified Equations of the First Degree.*

A. Expressions of the form $x = a \pm b \pm \cdots$, or $x = \frac{a \pm b \pm c \pm \cdots}{n}$.

1. To divide a triangle into two parts having equal perimeters by drawing a line from one vertex.

2. To divide a parallelogram by a line drawn from one vertex into two parts such that the perimeter of one shall exceed that of the other by a given length a.

3. To take equal lengths from two sides of a given triangle, measured from their intersection, so that the sum of the remainders shall be equal to the third side.

4. To describe a circle about each of three given points so that each circle shall touch the other two externally.

5. To draw parallels at equal distances, respectively, from the sides of a given rectangle, so that they shall form another rectangle of given perimeter.

6. To divide the greater side of a given rectangle into two segments, so that the difference of their squares shall be equal to the area of the rectangle.

7. To divide a given trapezoid into two equivalent quadrilaterals by drawing a line parallel to one of the legs.

8. To divide one side of a given triangle into two parts such that their difference shall be equal to one-third of the sum of the other two sides.

B. Expressions of the form $x = \frac{ab}{c}$.

9. Upon one side of a given triangle, to construct a rectangle equal in area to the product of the other two sides.

10. Given a point P in the side AB of the rectangle $ABCD$; through P to draw a line meeting the side BC produced in a point X, so that the triangle PBX shall be equivalent to the rectangle.

Given a triangle ABC; to draw a line XY parallel to AB (X in AC, Y in BC) so that one of the following conditions shall be satisfied:

11. $XY = BY$.

12. $CX = BY$.

13. $XY = AX + BY$.

14. $XY = BY - AX$.

15. $XY = CX + BY$.

16. $XY = CX - BY$.

17. $AX + BY = d$.

18. $AX - BY = d$.

19. $CX + BY = d$.

20. $BY - CX = d$.

21. $AB + XY = CX + CY$.

22. $AB - XY = CX + CY$.

23. Through a given point in one side of a triangle, to draw a line cutting off a triangle equal in area to one-third of the given triangle.

24. Upon a given line are four points in the order A, C, D, B; to find in the line a point X such that $AX : BX = DX : CX$.

25. Upon a given line as one side, to construct a rectangle equivalent to a rhombus whose diagonals are equal, respectively, to two given lines.

C. Expressions of the form $x = \frac{a^2}{b}$.

26. Upon a given side, to construct a rectangle equivalent to a given square.

Given a triangle ABC; to draw a line XY parallel to AB, so that one of the following conditions are satisfied:

27. $XY^2 = CX \times AC$. (Let $x = XY$.)

28. $XY = CX - AX$. (Let $x = CX$.)

29. $XY = AX - CX$. (Let $x = CX$.)

30. To produce a given chord of a circle, so that the tangent drawn from its extremity shall have a given length.

31. To transform a given square into a triangle, having a given altitude and a given angle at the base.

32. To produce a given line AB to a point X, such that the product $AB \times AX$ shall be equal to a given square.

D. Expressions of the form $x = \frac{a^2 \pm b^2}{nc}$

33. To construct a rectangle, given one side and the sum of the diagonal and the other side.

34. To construct a rectangle, given one side and the difference of the diagonal and the other side.

35. Upon one side of a given triangle, to construct a rectangle equivalent to the sum of the squares of the other two sides.

36. Given a point P in a diameter of a given circle (centre O); to produce the diameter to a point X, such that the tangent drawn from X to the circle shall be equal to PX. (Let $a = OP$, $x = OX$.)

GROUP II. *Unclassified Equations of the First Degree.*

To divide a given line into two segments such that:

37. The difference of their squares shall be equal to b^2.

38. Their ratio shall be equal to that of two given squares.

39. Their ratio shall be equal to that of two given rectangles.

Given a point C in a line AB; to find in this line a point X such that:

40. $AX^2 = BX \times CX$.

41. $AX : BX = AC^2 : BC^2$.

Given a triangle ABC; to draw a line XY parallel to BC, so that one of the following conditions shall be satisfied:

42. $XY^2 = AC \times CY$.

43. $XY^2 = CY \times BY$.

44. $XY^2 = AX \times CY$.

45. $AX^2 - AY^2 = XY \times BC$.

46. So that the two parts of the triangle ABC shall have equal perimeters.

47. So that the perimeter of the triangle AXY shall be equal to half that of the trapezoid $BCYX$.

To inscribe in a given triangle:

48. A square.

49. A rhombus, given one of its angles.

50. A rectangle, given the ratio of its sides.

51. A rectangle, given its perimeter.

52. A rectangle with one diagonal parallel to one side of the triangle.

53. A parallelogram of given perimeter, and having one angle in common with the triangle.

54. To divide a trapezoid by a line parallel to the bases into two parts having equal perimeters.

55. To circumscribe a trapezoid about a given circle, given the lengths of the two legs. Find also the area of the trapezoid in terms of the given parts.

To construct a right triangle, given:

56. a, ρ. **59.** $a+b+c, h$. **61.** $c+a, b$.

57. $c-a, \rho$. **60.** $a+b-c, h$. **62.** $c-a, b$.

58. $a+b, \rho$. (In 59–62 compute the hypotenuse c.)

In one side of a given triangle, to find a point such that the perpendiculars dropped from this point to the other two sides:

63. Shall have a given sum.

64. Shall be in a given ratio.

65. In a given square to inscribe five equal circles, so that the middle circle shall touch the four others, and each of these four shall touch two adjacent sides of the square.

66. Given P in L, and Q not in L; to find a point X in L, so that $PX+QX=a$.

67. Given P in L, and a second line L'; to find a point X in L, so that the sum of its distances from P and from L' shall be equal to a given length a.

68. Given a rectangle; to construct a square such that the perimeters of the two figures shall have the same ratio as their areas.

69. To draw a line parallel to one side of a given triangle which shall cut off a trapezoid having a given perimeter.

GROUP III. *Pure Quadratic Equations.*

70. Given C in AB; to find X in AB, so that $AX^2 = AB \times AC$.

71. In a triangle ABC, to draw a line XY between AC and BC, so that the triangle CXY shall be isosceles and equal in area to half the triangle ABC.

72. To transform a given parallelogram into a rhombus, one angle remaining unchanged.

73. To construct a circle which shall pass through two given points, and shall touch a given line.

To divide a given triangle by a line parallel to one side into:

74. Two equivalent parts.

75. Two parts in the ratio 2 : 3.

76. Within a given circle to construct five equal squares, so that each of the four outer squares shall have two vertices in the circumference, and one side in common with the fifth square.

77. To construct a right triangle, given c, and $a = 2b$.

78. To construct a square which shall be to a given square as 3 : 2.

To construct two lines (x and y) given:

79. Their ratio and their product.

80. Their ratio and the difference of their squares.

81. Their sum and the difference of their squares.

82. To transform a given square into a rectangle whose sides are as 3 : 4.

Given a length a, a circle, and an exterior point P; to draw a secant PAB cutting the circumference in A, B, so that:

83. $PA \times AB = a^2$. **84.** $PB \times AB = a^2$.

85. Given a length a, a circle, and an interior point P; to draw through P a chord meeting the circumference in A, B, so that $AB \times PB = a^2$.

To divide a given line AB in X so that:

86. $AB^2 - AX^2 = b^2$. **87.** $AB^2 + BX^2 = b^2$.

To divide a given trapezoid, by a line parallel to the bases, into:

88. Two equivalent parts.

89. Two parts in a given ratio $m : n$.

90. To cut from a given triangle, by a line parallel to one side, a trapezoid of given area.

GROUP IV. *Complete Quadratic Equations Classified.*

A. Equations of the form $x^2 \pm ax - a^2 = 0$.

91. To divide a given line in extreme and mean ratio.

92. To divide a given line into two parts, such that the smaller shall be a mean proportional between the greater and the difference of the two.

93. Given a circle and a chord; to produce the chord to a point X, so that the tangent drawn from X shall be equal to the chord.

94. In the triangle ABC to draw XY parallel to AB, so that $AB \times AX = XY \times CX$.

95. Upon a given line as hypotenuse to construct a right triangle in which the greater leg shall be a mean proportional between the hypotenuse and the other leg.

To produce a line AB to a point X, so that:

96. $AB^2 = AX \times BX$. **97.** $BX^2 = AB \times AX$.

B. Equations of the form $x^2 \pm ax - b^2 = 0$.

98. To transform a given square into a rectangle, having given the difference of its sides.

Given the point C in the line AB; to find the point X in AB, so that:

99. $AB \times BX = AX \times CX$.

100. $AX^2 + BX^2 - CX^2 = AC^2$.

101. Given a circle and a chord; to produce the chord to a point X, so that the tangent drawn from X shall have a given length.

102. To construct a circle which shall touch a given line and cut a chord of given length from another given line.

103. Through a given point within a given circle to draw a chord so that the difference of the two segments shall be be equal to a given line.

104. To divide a given line into two parts, so that the square of one part shall be equal to half the square of the other part.

105. To construct a right triangle, given $a - b$, h.

C. Equations of the form $x^2 \pm ax - bc = 0$.

106. To divide a given line into two parts, so that the square of one part shall be equal to the product of the other part and a given length.

Given C in AB; to find X in AB, so that:

107. $CX^2 = AC \times BX$. **108.** $AX^2 = AB \times CX$.

109. To transform a given square into an isosceles triangle in which the difference of the base and altitude is given.

110. A semi-circumference is divided by a line perpendicular to the diameter into two unequal parts; to construct a circle which shall touch the diameter, the perpendicular, and the smaller part of the semi-circumference.

D. Equations of the form $x^2 \pm ax + b^2 = 0$.

NOTE. If $b > \frac{1}{2}a$, the solution is impossible. If $b = \frac{1}{2}a$, $x = \pm \frac{1}{2}a$. Therefore, $b = \frac{1}{2}a$ is the *maximum* value of b, if b be regarded as a variable quantity; and $a = 2b$ is the *minimum* value of a, if a be regarded as a variable quantity.

111. To divide a given line into two parts the product of which shall be equal to the square of a given line. When is this product a maximum?

112. To construct a rectangle, given its perimeter and its area. Of all rectangles of given perimeter which has the greatest area? Of all rectangles of given area which has the least perimeter?

113. To inscribe in a given equilateral triangle another equilateral triangle one-half as large.

114. Given C in AB; to find a point X in AB, such that $CX^2 = AX \times BX$.

115. In a given isosceles right triangle to inscribe a rectangle having a right angle in common with the triangle, and equivalent to a given square.

E. Equations of the form $x^2 \pm ax + bc = 0$.

NOTE. As regards maxima and minima, see note under D.

116. To divide a given line into two parts, so that their product shall be equal to the area of a given rectangle.

117. Given C in AB; to find a point X in AB between B and C, so that $AX^2 = BX \times CX$.

118. Given C in AB; to find a point X in AB between A and C, so that $AB : AX = BX : CX$.

119. To construct a rectangle equivalent to a given rectangle, and having a perimeter equal to that of another given rectangle.

120. To construct a right triangle, given $a+b$, c.

121. To construct a right triangle, given $a-b$, c.

122. To divide the greater side of a given rectangle into two parts, so that the sum of their squares shall be equal to the area of the rectangle.

123. To transform a given square into an isosceles triangle in which the sum of the base and the altitude is given.

124. In a given square to inscribe another square the side of which has a given length.

125. To transform a given triangle into a rectangle having the same perimeter.

ANSWERS

TO THE

NUMERICAL EXERCISES IN CHAPTERS III., IV., V.

CHAPTER III. § 14.

1. 61.44 ft.
2. 16 ft.
3. 35.49 ft.
4. 122 ft., 183 ft.
5. No.
6. 4 and 5, 4.5 and 7.5, 6.43 and 8.57.
7. $\frac{ac}{a+b}$, $\frac{bc}{a+b}$; $\frac{c}{2}$.
8. 2 : 1.
9. 60, 56.25.
10. $AB = 8$ ft., $AD = 10.66$ ft., $DE = 5.33$ ft.
11. 16.25 in.; 8.75 in.
12. $EF = 2287$ ft.
13. $\frac{ah}{a-b}$.
14. 90 ft.
15. 320 ft.
16. $4\frac{1}{2}$ miles.
17. 8.57 ft.
18. 26 ft.
19. 20 ft.
20. $\frac{a}{2}\sqrt{3} = 0.866 \times a$.
21. $x = 4.8$ ft., $y = 6.4$ ft.
22. $x = 3.846$ ft., $y = 22.154$ ft., $h = 9.231$ ft.
23. $x = \frac{a^2}{\sqrt{a^2+b^2}}$, $y = \frac{b^2}{\sqrt{a^2+b^2}}$, $h = \frac{ab}{\sqrt{a^2+b^2}}$.
24. $x = 0.6$, $y = 0.8$.
25. 3, 4, 5. The second solution $(-1, 0, +1)$ does not belong to this question.
26. 24, 32.
27. 33, 44, 55.
28. 12 ft.
29. 2.238, 3.059.

30. 3 in.
31. $r\sqrt{3}$, 120°.
32. 12 in.
33. $\sqrt{r^2+r'^2}$.
34. $\sqrt{200} = 14.142$ in.
35. Each tangent = 8 in.; chord of contact = $9\frac{6}{5}$ in.
36. $r = \frac{a^2+b^2}{2a}$; $r = a$, and ab is a diameter.
37. Obtuse, since $8^2 + 9^2 < 13^2$. (No. 163.)
38. $\frac{a^2}{2\sqrt{a^2-b^2}}$.
39. $\sqrt{ab+c^2}$.
40. See text.
41. 7 in. and $4\frac{1}{2}$ in.
42. 10 in. and 2 in.
43. 14.14 ft.
44. $r\sqrt{10}$.
45. $3\sqrt{5}$ in.

CHAPTER IV. § 20.

1. 5 acres 1400 sq. yds.
2. 2.25^{qm}.
3. 39 sq. ft. 9 sq. in.
4. 648 sq. yds.
5. 220 yds.
6. 200 rds.
7. $60.48.
8. $166.50.
9. 7.789 yds.
10. 49 sq. ft. 54 sq. in.
11. 7 acres 3920 sq. yds.
12. 8748 sq. yds.
13. 11 yds.
14. 97 yds. $2\frac{1}{3}$ ft.
15. 288 sq. ft.
16. 25 sq. ft.
17. 225 sq. yds.
18. The rectangle, 8^{m}.
19. 18 ft. and 12 ft.
20. 30 ft. and 10 ft.
21. 12 ft. and 5 ft.
22. 42 ft. and 18 ft.
23. 16 ft. and 12 ft.
24. 2700^{qm}.
25. 60 ft. and 40 ft.
26. 4 ft.
27. 7168 tiles.
28. 2 ft. 4 in.
29. $568.32.
30. Gained $3518.
31. No difference.
32. 11,250 sq. ft.
33. 96^{qm}.
34. 108 sq. ft.

35. $16\frac{7}{8}$ sq. ft.
36. 20 ft.
37. No; 1 : 2.
38. 1 : 2.
39. 17 ft. and 20 ft.
40. 33.294^{m}.
41. 75 yds., 72 yds.
42. 1164 sq. ft. 54 sq. in.
43. 1 : 3.
46. $a : a' = h' : h$.
47. They are equal.
48. 448 sq. miles.
49. 30 sq. ft.
50. 162 sq. ft.
51. 100 sq. ft.
52. 30 sq. ft.
53. 5.196 sq. in.
54. 432 sq. ft.
55. 1.732 sq. ft.
56. 692.82 sq. ft.
57. 332.544 sq. ft.
58. 3 : 4.
59. 200 ft.
60. 5240 sq. ft. nearly.
61. 44.7213^{m}.
62. 150 sq. ft.
63. 150 sq. ft.
64. 216 sq. ft. and 384 sq. ft.
65. 70.9298^{qm}.
66. 48.441^{m} and 12.386^{m}.
67. Base 70 ft., altitude 25 ft.
68. $147\frac{7}{8}$ sq. ft.
69. 5460 sq. ft.
70. 420 sq. ft. 71. 0.38 ft.

72. Let $s = \frac{1}{2}(a+b+c)$; then

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

$$\rho = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

73. 16.8 ft.
74. 7.49^{qm}.
75. 1050 sq. ft.
76. 300 sq. ft.
77. 20 ft.
78. (i.) $13\frac{1}{2}$ ft. and $16\frac{1}{2}$ ft.
(ii.) 12 ft. and 18 ft.
79. $1600.
80. 208 sq. ft.
81. 4 sq. ft.
82. 28 ars; $33\frac{1}{3}$ ars.
85. $417.60.
86. 4896 sq. ft.
87. 5000 sq. ft.
88. 6 acres 6440 sq. ft.
89. 80 ft.

90. 160 ft.
91. 40 ft.
92. $18\frac{6}{13}$ ft.
93. 40 ft.
94. $27\frac{1}{2}$ ft.
95. 18 ft.
96. $3\frac{7}{11}$ ft.
97. 12.7279 ft.
98. 7.8964 ft.
99. 18.2363 ft.
100. 14.7418 ft.
101. 8.658 ft.
102. 10.528 ft.
103. 23.59 ft.
104. 62.161 yds.
105. 80 ft.
106. 30 ft. and 45 ft.
107. 12 ft. and 8 ft.
108. (i.) $\frac{1}{6}$ and $\frac{5}{6}$; (ii.) $\frac{1}{2}$ and $\frac{1}{2}$; (iii.) $\frac{1}{4}$ and $\frac{3}{4}$; (iv.) $\frac{1}{25}$ and $\frac{24}{25}$; (v.) $\frac{1}{mn}$ and $\frac{mn-1}{mn}$; (vi.) $\frac{1}{n^2}$ and $\frac{n^2-1}{n^2}$.
109. $\frac{1}{2}$ and $\frac{1}{3}$, or $\frac{2}{3}$ and $\frac{3}{5}$, etc.
110. 60 ft.
111. One line to the adjacent side dividing it in the ratio 1 : 2, the other line to the opposite vertex.
112. $AE = \frac{8}{15}AC$, $BF = \frac{12}{25}BC$.
113. $AD = \sqrt{\frac{1}{2}} \times AB$.
114. $AD = \sqrt{\frac{1}{3}} \times AB$.
115. $AD = \sqrt{\frac{2}{3}} \times AB$.
116. $AD = \sqrt{\frac{m}{m+n}} \times AB$.
117. $AE = 21.21$ ft.
118. $\frac{1}{4}$.
119. 5 : 1.
120. $\frac{7}{12}$.
121. 2 : 1.
122. $AF = 3$ ft. 9 in.
123. $DX : XB = 8 : 17$.
124. 9.87 ft.; $0.564 \times h$.
125. Upper part $= \frac{h}{18}(a + 5b)$;
Middle part $= \frac{h}{6}(a + b)$;
Lower part $= \frac{h}{18}(5a + b)$.

126. 1250 sq. ft.

127. 40 sq. ft.

128. 60 ft.

129. $\frac{a}{\sqrt{2}}$.

130. $\frac{4\sqrt{3}}{\sqrt{5}} = \frac{4}{5}\sqrt{15}$.

131. $2\sqrt{14}$.

132. 21.6333 ft.

133. 5 ft.

134. 65 ft.

135. 2 : 1.

136. (i.) 4 : 1; (ii.) 2 : 1.

137. 4 : 3.

138. $31\frac{1}{3}$ ft., $52\frac{2}{3}$ ft.

139. 9 ft.

140. Bases, $a + \frac{\sqrt{a^4 - b^4}}{a}$, and $a - \frac{\sqrt{a^4 - b^4}}{a}$;
Each leg $= a$.

141. $F = \frac{1}{2}a\sqrt{c^2 - a^2}$.

142. $F = \frac{a^2 h}{2\sqrt{a^2 - h^2}}$.

143. $F = \frac{h(q^2 + h^2)}{2q}$.

144. $F = \frac{1}{2}c\sqrt{pq}$.

145. $F = \frac{1}{4}a^2\sqrt{3}$.

146. $F = \frac{1}{3}a^2\sqrt{3}$.

147. $F = \frac{1}{4}c\sqrt{4a^2 - c^2}$.

148. $F = h\sqrt{b^2 - h^2}$.

149. $F = \frac{h^2 h_a}{\sqrt{4h^2 - h_a^2}}$.

150. $a = 2\sqrt{\frac{F\sqrt{3}}{3}}$, $h = \sqrt{F\sqrt{3}}$.

151. $\frac{am}{h + m}$.

152. (i.) $\frac{1}{2}(hm + an + mn)$.
(ii.) $\frac{1}{2}(hm + an - m)n$.
(iii.) $\frac{1}{2}(hm - an - mn)$.
(iv.) $\frac{1}{2}(an - hm - mn)$.

153. (i.) $F = (3 - 2\sqrt{2})\,s^2$.
(ii.) $F = (3 + 2\sqrt{2})\,d^2$.

154. mn, $(a - n)m$,
$(b - m)n$, $(a - n)(b - m)$.

155. $f = \sqrt{2(a^2 + b^2) - g}$.

156. $x = \frac{ab}{2(a + b)}$.

157. $\frac{1}{2}(\sqrt{f^2 + 2F} \pm \sqrt{f^2 - 2F})$.

158. $\frac{1}{2}(\sqrt{4a^2 + 2F} \pm \sqrt{4a^2 - 2F})$.

159. $x = \sqrt{\frac{ma^2 + (n + p)c^2}{m + n + p}}$; $y = \sqrt{\frac{(m + n)a^2 + pc^2}{m + n + p}}$.

CHAPTER V. § 23.

22. $c = 2a$, $c = 2r$, $c = \frac{2r\sqrt{3}}{3}$.

23. $6\sqrt{2-\sqrt{3}} = 3.1048$ sq. ft.

24. $12r^2(2-\sqrt{3})$.

25. 8 : 3.

26. $\frac{1}{8}$.

27. 70.53 ft.

28. $2r(\sqrt{2}-1)$.

29. $\frac{c\sqrt{10+2\sqrt{5}}}{4}$.

30. 4.902 ft.

31. 13 ft.

32. 10 acres 1843 sq. yds., nearly.

33. $216.

34. Octagon, by 754.4 sq. ft.

36. 1 ft. 10 in.

37. 4 ft. 1 in.

38. 120 revolutions.

39. 3 ft. $9\frac{9}{11}$ in.

40. $17\frac{1}{2}$ ft.

41. 220 ft.

42. 11 in.

43. 45°.

44. $57\frac{3}{11}°$.

45. $57\frac{3}{11}°$.

46. 14 ft. $7\frac{1}{2}$ in.

47. 4 ft. $4\frac{1}{2}$ in.

48. $16\frac{1}{2}$ ft.

49. 7 ft. 4 in.

50. 30.86m.

51. 6369.42k.

52. 1 : 2.

53. 4 ft.

54. 1 sq. ft. 10 sq. in.

55. 203 acres 3080 sq. yds.

56. 192.249 yds.

57. 352 yds.

58. 308 sq. ft.

59. 49.638 ft.

60. 154 sq. ft.

61. $107.86.

62. 2.984 yds.

63. 39.597 ft.

64. 1050 sq. in.

65. 28.284 in.

66. 5544 sq. ft.

67. 240 sq. ft. 90 sq. in.

68. Square, 1089 sq. ft.;
Circle, 1386 sq. ft.;
Triangle, 838.312 sq. ft.

69. 15 ft.

70. $\sqrt{r^2 - r'^2}$.

71. $4\pi : 3\sqrt{3}$.

72. 1 : 4.

73. Circle, 2.122$^{\text{sq m}}$;
Triangle, 0.877$^{\text{sq m}}$.

74. \$376.24$\frac{1}{2}$.

75. 5.196 ft., 7.348 ft.

76. \$2461.85.

77. 41$\frac{2}{7}$ ft.

78. 4 ft.

79. 1 ft. 3 in.

80. 2 ft. 1 in.

81. 102 ft.

82. 240 ft.

83. 14.277 ft.

84. 78$\frac{4}{7}$ sq. ft.

85. 33.312 in.

86. 107.274 ft., 225.868 ft.

87. 222.562 ft.

88. 616 sq. ft.

89. 35 ft., 92 ft.

90. 578.6625 sq. ft.

91. 546.175 sq. ft.

92. 442$\frac{3}{16}$ sq. ft.

93. 342$\frac{2}{16}$ sq. ft.

94. 2066$\frac{2}{3}$ sq. ft.

95. $\frac{r^2(4\pi - 3\sqrt{3})}{12}$.

96. $\frac{r^2(2\pi - 3\sqrt{3})}{12}$.

97. $\frac{r^2(\pi - 2)}{4}$.

98. $\frac{r^2(\pi - 2\sqrt{2})}{8}$.

99. $\frac{r^2(8\pi - 5\sqrt{10 + 2\sqrt{5}})}{40}$.

100. $\frac{r^2(4\pi - 5\sqrt{10 - 2\sqrt{5}})}{40}$.

101. 3π sq. ft.

102. 12.092 sq. ft.

103. $\frac{10\pi + 3\sqrt{3}}{12}$.

104. $\frac{4\pi - 3\sqrt{3}}{8\pi + 3\sqrt{3}} = 0.36968$.

105. 4.43$^{\text{sq m}}$ and 149.5$^{\text{sq m}}$.

106. 27° 7′ 18″.

107. 28.2 ft. and 45.13 ft.

108. 7$\frac{1}{2}$ ft.

109. 2 ft.

110. 2.414 ft.

111. $\frac{\pi}{4}(a^2 + b^2)$.

112. $\frac{r^2}{2}$.

113. $r^2\left(\sqrt{3} - \frac{\pi}{3}\right)$.

114. $r^2\left(\sqrt{3} - \frac{\pi}{3}\right)$.

115. (i.) $\pi r^2(3 - 2\sqrt{2})$; (ii.) $\frac{\pi r^2}{2}(3\sqrt{2} - 4) + \frac{r^2}{2}(4\sqrt{2} - 6)$.

116. $\frac{a^2(\pi - 2)}{2}$.

EXERCISES IN SOLID GEOMETRY.

NOTE. When π occurs as a factor, take $\pi = 3\frac{1}{7}$, if English units are used, and $\pi = 3.1416$ if Metric units are used.

The weight of 1 cubic foot of water is to be taken as 1000 oz.

§ 1. PLANES.

1. How many perpendiculars in space can be erected at a given point in a given line? How do they lie relatively to one another?

2. How many perpendiculars in space can be dropped from a given point to a given line?

3. If two lines in space are both perpendicular to a given third line, are they necessarily parallel to each other?

4. Can two lines not in the same plane both be perpendicular to a given plane?

5. If two lines in space are both perpendicular to a given plane, are they necessarily parallel to each other?

6. If two lines in space are parallel to a given plane, are they necessarily parallel to each other?

7. Can two vertical planes intersect? two horizontal planes?

8. Through a vertical line how many vertical planes can be passed? how many horizontal planes?

9. Through a horizontal line how many horizontal planes can be passed? how many vertical planes?

10. The distance from a point A to a point B, situated in a given plane, is 20 in., and the distance from B to the foot of the perpendicular dropped from A to the plane is 16 in.; find the distance from A to the plane.

11. The distances of two points, A, B, from a given plane are 5 in., 11 in., respectively, and the distance between the feet of the perpendiculars dropped from A, B to the plane is 8 in.; find the distance from A to B.

12. If the line drawn from a point to a plane, and making an angle of 45° with the plane, is 10 in. long, what is the distance from the point to the plane?

13. Through a point 6 in. from a plane a line is drawn to the plane, making the angle 30° with the plane; find the length of this line.

14. Through a point A above a given plane a perpendicular AO is drawn to the plane; about O as centre a circumference is described, and through a point B in this circumference a tangent BC is drawn. Given, $AO=12$ in., $OB=7$ in., $BC=15$ in.; find AC.

15. A line makes, with a plane, an angle of 24°; what are the least and the greatest angles which it makes with lines drawn through its foot in the given plane?

16. Describe different relative positions which three planes may have.

17. The line AB meets three parallel planes in the points A, E, B; and the line CD meets the same planes in the points C, F, D. If $AE=6$ in., $BE=8$ in., $CD=12$ in., compute CF and FD.

18. What is the locus of the points in a plane which are at a given distance from a given point not in the plane?

What is the locus in space of all points:

19. Equidistant from two given points?

20. Equidistant from three given points?

21. Equidistant from two given parallel lines?

22. Equidistant from three given parallels not in the same plane?

23. Equidistant from two given intersecting lines?

24. Equidistant from the three edges of a trihedral angle?

25. At a given distance from a given plane?

26. Equidistant from two given parallel planes?

27. Equidistant from two given intersecting planes?

28. Equidistant from the three faces of a trihedral angle?

29. What is the locus of all lines drawn through a given point, and parallel to a given plane?

30. What is the locus of points in a given plane which are equidistant from two given points not in the plane?

31. What is the locus of points equidistant from two given planes, and also equidistant from two given points?

32. What position relative to a given plane has a line if its projection on the plane is equal to its own length? What position if its projection is a point?

33. A line 4 feet long makes, with a plane, the angle 45°; find its projection on the plane.

34. If the projection of a line upon a plane is equal to half the length of the line, what is the angle of inclination of the line to the plane?

§ 2. The Prism.

1. One edge of a cube $= a$; find the surface, the volume, and the length of a diagonal.

2. Find one edge and the volume of a cube, if the surface = 96 sq. ft. Also find the same, if the surface $= S$.

3. Find the edge and the surface of a cube, if the volume = 1000 cub. ft. Also if the volume $= V$.

4. How many square meters of surface require to be cemented, in constructing a cubical reservoir which will hold 8000^{kg} of water?

5. Find the dimensions of a cubical block of marble which weighs 2700^{kg}. Specific gravity of marble = 2.7.

6. Find the volume of a cube, if a diagonal in one of its faces $= a$.

7. How much lead is used in lining the bottom and sides of a cubical vessel that holds 729 cub. ft. of water?

8. If a cubical vessel requires 320 sq. ft. of lead for lining the bottom and sides, how many cubic feet of water will it hold?

9. A plumber is ordered to make a cubical vessel which will hold 2 tons of water. What must be the length of one edge?

10. A cellar 12 ft. long and 9 ft. wide is flooded to a depth of 4 in. Find the weight of the water.

11. What is the weight of the air in a room 5^m long, 4^m wide, and 3.2^m high, if one liter of air weighs 1.293^g?

12. How much lead will be required to line a cistern, open at the top, which is 4 ft. 6 in. long, 2 ft. 8 in. wide, and contains 42 cub. ft.?

13. How many bricks are required to build a wall 90 ft. long, 8 ft. high, and 18 in. thick, if the bricks are 9 in. long, $4\frac{1}{2}$ in. wide, and 3 in. thick?

14. A book is 8 in. long, 6 in. wide, and $1\frac{1}{4}$ in. thick. Find the depth of the box whose length and breadth are 3 ft. 4 in. and 2 ft. 6 in., that it may contain 400 such books.

15. An open cistern is made of iron 2 in. thick. The inner dimensions are: length, 4 ft. 6 in.; breadth, 3 ft.; depth, 2 ft. 6 in. What will the cistern weigh (i.) when empty? (ii.) when full of water? Specific gravity of iron = 7.2.

16. An open cistern 6 ft. long and $4\frac{1}{2}$ ft. wide holds 108 cub. ft. of water. How many cubic feet of lead will it take to line the sides and bottom, if the lead is $\frac{1}{8}$ in. thick?

17. Rain has fallen to the depth of half an inch; how many cubic feet of water have fallen on an acre?

18. The three dimensions of a rectangular parallelopiped are a, b, c; find the surface, the volume, and the length of a diagonal.

19. The volume of a parallelopiped $= V$, and the three dimensions are as $m : n : p$; find the dimensions.

Find the lateral surface and the volume of the following regular prisms, if in each case the height $= h$, and one side of the base $= a$:

20. Triangular prism.

21. Quadrangular prism.

22. Hexagonal prism.

23. Octagonal prism.

24. Find the volume of a right triangular prism, if its height is 14 in., and the sides of the base are 6, 5, and 5 in.

25. Find the volume of a right prism 14 ft. high, whose bases are squares, each side measuring 1 ft. 6 in.

26. The base of a right prism is a rhombus, one side of which = 10 in., and the shorter diagonal = 12 in. The height of the prism = 15 in. Find the entire surface and the volume.

27. Find the volume of a right hexagonal prism whose height is 10 ft., each side of the hexagon being 10 in.

28. Find the volume of a right prism 32 ft. long, if its ends are trapezoids, the parallel sides of which are 12 ft. and 8 ft., and the perpendicular distance between them is 6 ft.

29. Find the volume of a right prism, if the height = 3^m, and the base is a square the diagonal of which = 2^m.

30. How many cubic inches of mahogany will be required to veneer the top of a table in the shape of a regular hexagon, each side of which measures 2 ft., the veneer being $\frac{1}{8}$ of an inch thick?

31. How many cubic feet of stone are required to build a dam 1000 ft. long, 20 ft. high, 10 ft. wide at the bottom, and 4 ft. wide at the top?

32. The wall of China is 1500 miles long, 20 ft. high, 15 ft. wide at the top, and 25 ft. wide at the bottom; how many cubic yards of material does it contain?

33. The distance around a reservoir in the shape of a regular hexagon is 360 ft. If the average monthly loss from evaporation amounts to a layer of water 2 in. deep, how many cubic feet of water must be supplied monthly to maintain the water at a constant level?

34. Given the volume V and the height h of a regular hexagonal prism, find one side a of the base.

§ 3. The Cylinder of Revolution.

1. How many square feet of sheet iron are required to make a funnel 2 ft. in diameter and 40 ft. long?

2. A right cylinder is 10 ft. high, and measures 7 ft. 4 in. around the base. Find the convex surface and the volume.

3. Find the radius of the base of a right cylinder if the volume = 1540 cu. in., and the height = 10 in.

4. Find the height of a right cylinder if the volume = 3080 cub. ft., and the radius of the base = 7 ft.

5. I wish to have made a cylindrical pail 14 in. high, and holding exactly 4 cub. ft.; what must be the radius of the base?

6. If the total surface of a right cylinder closed at both ends is a, and the radius of the base is r, what is the height of the cylinder?

7. If the lateral surface of a right cylinder is a, and the volume is b, find the radius of the base and the height.

8. By how much is the volume of a right cylinder increased if the radius of the base is doubled? if the height is doubled? if both are doubled?

9. What will it cost to dig a well 3 ft. in diameter and 30 ft. deep, at $4.00 per cubic yard of earth thrown out?

10. How many cubic yards of earth must be removed in constructing a tunnel 100 yds. long, whose section is a semicircle with a radius of 10 ft.?

11. If the diameter of a well is 7 ft., and the water is 10 ft. deep, how many gallons of water are there, reckoning 7½ gals. to the cubic foot?

12. If a cubic foot of brass is drawn into a wire $\frac{1}{11}$ of an inch in diameter, find the length of the wire.

13. Find the weight of a cylindrical iron shell 1 in. thick and 2 ft. long, the inner radius being 7 in. Specific gravity of iron = 7.2.

14. A rectangular sheet of tin 44 in. long and 14 in. wide is bent so as to form a cylindrical surface 14 in. high. Find the area of the base, and the volume.

15. Find the weight of mercury contained in a cylindrical vessel, if the diameter of the base = 2^{dm} and the depth of the mercury = 4^{dm}. Specific gravity of mercury = 13.6.

16. A cylindrical vessel holding 20 liters has a height just equal to the diameter of the base; find its dimensions.

17. The volume of a right cylinder = 340 liters; find the lateral surface, if the height is equal to twice the diameter of the base?

18. The height of the zinc vessel holding 1 liter is double the diameter of the base; the thickness of the metal is 0.5^{cm}, and the specific gravity of zinc is 7.19; find the weight of the vessel.

19. How many cubic feet of lead will be required to line the curved surface and bottom of an open cylindrical vessel whose depth is 7 ft., and which will hold 198 cub. ft. of water, if the lead is $\frac{1}{8}$ of an inch thick?

20. A marble column measures 7 ft. 4 in. in circumference, and its height is 15 ft.; find the expense of polishing the curved surface at \$1.50 per square foot; and also the additional expense if the whole surface is polished.

21. A cubic foot contains $7\frac{1}{2}$ gals. If 375 gals. are pumped out of a well 7 ft. in diameter, how many inches will the surface of the water fall in consequence?

22. What length of wire 0.08 of an inch thick can be made out of a cubic inch of metal?

23. How many revolutions of a roller 3 ft. in length and 18 in. in diameter would it take to go over a grass plot half an acre in extent?

24. About the convex surface of a right cylinder 2 ft. 4 in. in diameter, and 3 ft. 4 in. high, a cord $\frac{1}{8}$ of an inch in diameter is wound until the surface is completely covered. How many feet of cord are required?

25. A cast iron cylinder 3 ft. long and 7 in. in diameter is reduced in a lathe to a diameter of 6 in.; find the loss in weight. Specific gravity of iron = 7.2.

26. The piston of a pump is 16^{cm} in diameter, and moves through a space of 46^{cm}. How many liters of water are thrown out by 1000 strokes?

27. When a body is placed under water in a right cylinder 60^{cm} in diameter, the level of the water rises 30^{cm}; find the volume of the body.

28. How much will a brass cylinder weigh under water if the height is 64^{cm}, and the diameter of the base is 40^{cm}? Specific gravity of brass = 7.8. *The loss in weight of an immersed body is equal to the weight of the liquid displaced.*

29. If the circumference of the base of a right cylinder is c, and the height h, find the volume V.

30. Having given the total surface T of a right cylinder, in which the height is equal to the diameter of the base, find the volume V.

31. If the circumference of the base of a right cylinder is c, and the total surface is T, find the volume V.

32. If the volume of a right cylinder is V, and height h, find the total surface T.

33. If the volume of a right cylinder, in which the height is equal to the diameter of the base, is V, find the height h and the total surface T.

34. Having given the total surface T of a right cylinder, and the height h, find the diameter and the volume.

§ 4. The Pyramid.

1. Two pyramids standing on the same plane have the same height, and the areas of their bases are 120 sq. ft. and 180 sq. ft. respectively. A plane parallel to the plane of their bases cuts from the first pyramid a section of 70 sq. ft.; find the area of the section of the second pyramid.

2. A pyramid 15 ft. high has a base containing 169 sq. ft. At what distance from the vertex must a plane be passed parallel to the base so that the section may contain 100 sq. ft.?

3. The base of a pyramid contains 144 sq. ft. A plane parallel to the base and 4 ft. from the vertex cuts a section containing 64 sq. ft.; find the height of the pyramid.

4. A pyramid 12 ft. high has a square base measuring 8 ft. on a side. What will be the area of a section made by a plane parallel to the base and 4 ft. from the vertex?

5. Two pyramids standing on the same plane have the same height, 14 ft. The first has for base a square measuring 9 ft. on a side; the second a hexagon measuring 7 ft. on a side. What will be, in each pyramid, the areas of the sections made by a plane parallel to the bases and 6 ft. from their vertices?

6. The base of a regular pyramid is a hexagon of which the side measures 3 ft. Find the height of the pyramid if the lateral area is equal to ten times the area of the base.

Find the volume in cubic feet of a regular pyramid:

7. When its base is a square, each side measuring 3 ft. 4 in., and its height is 9 ft.

8. When its base is an equilateral triangle, each side measuring 4 ft., and its height is 15 ft.

9. When its base is a regular hexagon, each side measuring 6 ft., and its height is 30 ft.

Find the total surface in square feet of a regular pyramid:

10. When each side of its square base is 8 ft., and the slant height is 20 ft.

11. When each side of its triangular base is 6 ft., and the slant height is 18 ft.

12. When each side of its square base is 26 ft., and the perpendicular height is 84 ft.

Find the height in feet of a regular pyramid when its volume and its base are:

13. Volume 26 cub. ft. 936 cub. in., and each side of its square base 3 ft. 6 in.

14. Volume 20 cub. ft., and the sides of its triangular base 5 ft., 4 ft., and 3 ft.

15. The base edge of a regular pyramid with a square base measures 40 ft., the lateral edge 101 ft.; find its volume in cubic feet.

16. The volume of a regular pyramid with a square base is 200 cub. ft., its base edge 5 ft.; find the total surface in square feet.

17. The base edge of a regular triangular pyramid is 18^{cm}, the lateral edge 15^{cm}; find the slant height, the area of a lateral face, the height of the pyramid, the total surface, and the volume.

18. How many square feet of ground will the base of a pyramid cover which contains 102 cub. yds., 21 cub. ft., and the height of which is 45 ft.?

19. A cubic inch of gold weighs 11 oz. Avoir.; how many ounces will be used in making a solid gold ornament in the shape of a regular pyramid with a square base, if each side of its base is 3 in. and its height 6 in.?

20. Find the volume of a regular pyramid whose slant height is 12 ft. and whose base is an equilateral triangle inscribed in a circle having a radius of 10 ft.

21. Having given the base edge a, and the total surface T, of a regular pyramid with a square base, find the volume V.

22. The base edge of a regular pyramid whose base is a square is a, the total surface T; find the height of the pyramid.

23. The eight edges of a regular pyramid with a square base are equal in length, and the total surface is T; find the length of one edge.

24. Find the lateral edge of a regular pyramid with a square base, having given the height h and the total surface T.

§ 5. The Cone of Revolution.

1. The height of a right cone is 2^m, and the area of its base 1^{qm}; find the area of the section made by a plane parallel to the base, and 80^{cm} from the vertex.

2. The radius of the base of a right cone is 40 in., and the height of the cone is 8 ft. At what distance from the vertex must a plane be passed parallel to the base that the section may be a circle with a radius of 30 in.?

3. If a right cone is 4 ft. high, at what distance from the vertex must a plane be passed parallel to the base, in order that the section may be equal to $\frac{1}{3}$ of the base?

4. If the slant height of a right cone is a, and the area of the base b, find the area of a section made by a plane parallel to the base, at the distance c from the vertex.

5. The slant height of a right cone is 2 ft. At what distance from the vertex must the slant height be cut by a plane parallel to the base, in order that the lateral surface may be divided into two equivalent parts?

6. The height of a right cone is equal to the diameter of its base; find the ratio of the area of the base to the lateral surface.

7. The area of the convex surface of a right cone is 99 sq. ft., and the radius of its base is 2 ft. $7\frac{1}{2}$ in.; find its slant height.

8. The slant height of a right cone, whose convex surface measures 2310 sq. ft. is 35 ft.; find its height.

9. How many cubic feet of lead will be required to cover a conical spire which measures 35 ft. round the base, and has a slant height of 30 ft., if the lead is $\frac{1}{12}$ of an inch thick?

10. Find the expense of covering a conical spire which measures 40 ft. round the base, and has a slant height of 30 ft., with lead $\frac{1}{8}$ of an inch thick, if one cubic inch of lead weighs $6\frac{1}{2}$ oz., and lead is worth 8 cents a pound.

11. How much canvas is required for a conical tent, the altitude of which is 8 ft., and the diameter of the base is 7 ft.?

12. A conical tent, whose slant height is 12 ft., requires 132 sq. ft. of canvas to cover it; find how much ground the tent covers.

13. What will be the cost of painting a conical spire whose slant height is 118 ft., and whose circumference at the base is 64 ft., at 16 cents a square yard?

14. What length of canvas $\frac{3}{4}$ of a yard wide is required to make a conical tent 12 ft. in diameter and 8 ft. high?

15. The slant height of a right cone is equal to the diameter of the base. If the slant height is a, find the total surface.

16. What does the volume V of a cone become, if the height is doubled? If the radius of the base is doubled? If both the height and the radius of base are doubled?

17. A conical mound of earth measures 264 yds. round the base, and the length of its slope is 70 yds.; find the number of cubic yards in the mound.

18. Find the height of a conical vessel whose diameter at the base is 2 ft. 4 in., in order that its volume may be 27 cub. ft.

19. Find the volume of a right cone, the height of which is 15 ft., and the circumference of the base 14 ft.

20. The circumference of the base of a cone is $12\frac{1}{2}$ ft., and its height $8\frac{1}{4}$ ft.; find its volume.

21. A right cone, 3 ft. high and 2 ft. in diameter at the base, is placed on the ground, and sand is poured over it until it makes a conical heap 5 ft. high and 30 ft. in circumference; how many cubic feet of sand are used?

22. How many gallons are contained in a conical vessel the radius of whose base is 6 ft., and the slant height 10 feet? (A gallon is 231 cub. in.)

23. A conical wine-glass is 2 in. wide at the top, and 3 in. deep; how many cubic inches of wine will it hold?

24. The slant height of a right cone is 5 ft. A plane parallel to the base cuts the slant height at a distance of 2 ft. from the vertex. If the section made by this plane is a circle whose radius is $4\frac{1}{2}$ in., find the volume of the cone.

25. A right cone made of silver, whose height is twice the diameter of its base, weighs 2.5^{kg}; required the dimensions of the cone, the specific gravity of silver being 10.47.

26. A right cone is 4 ft. high. The section made by a plane parallel to the base at the distance of 1 ft. from the vertex contains 1 sq. ft.; find the volume of the cone.

27. The volume of a right cone is $\frac{1}{3}$ of a cubic meter, the radius of the base is 1^{m}; find the lateral surface.

When the slant height of a right cone is equal to the diameter of its base:

28. Find the diameter, if the volume is 25^{ccm}.

29. Find the volume, if the total surface is 1000^{qcm}.

30. Find the volume, if the total surface is T.

31. Given the total surface T of a right cone, and the radius r of the base; find the volume V.

32. Given the total surface T of a right cone, and the lateral surface S; find the volume V.

§ 6. Frustums of Pyramids and Cones.

1. How many square feet of tin will be required to make a funnel if the diameters of the top and bottom are to be 28 in. and 14 in. respectively, and the height 24 in.?

2. The slant height of a church spire in the shape of the frustum of a regular hexagonal pyramid is 20 ft., the length of each side of its base is 5 ft., and of its top 2 ft. How many square feet of lead will be required to cover its lateral faces and top?

3. Find the expense of polishing the curved surface of a marble column in the shape of the frustum of a right cone whose slant height is 12 ft., and the radii of the circular ends are 3 ft. 6 in. and 2 ft. 4 in. respectively, at 60 cents per square foot.

4. Find the lateral surface of the frustum of a right cone if the radii of the bases are 2^{m} and 6^{m}, and the height is 3^{m}.

5. Having given the lateral surface of the frustum of a right cone, 3454^{qdm}, and the radii of the bases, 1.42^{m} and 0.64^{m}, find the height of the frustum.

6. The height of a right cone is 15 ft., the radius of its base 2 ft. Find the lateral surface of the frustum made by a plane parallel to the base, and distant 12 ft. from the vertex.

7. Having given the slant height a of a right cone, and the radius r of the base, find at what distance from the vertex the slant height must be cut by a plane parallel to the base in order that the total surfaces of the two parts into which the cone is divided may be equal.

8. A round stick of timber is 20 ft. long, 3 ft. in diameter at one end, 2.6 ft. at the other; how many cubic feet does it contain?

9. Find the volume of the frustum of a right cone; given the radii of the circular ends, 3 ft. and 3 ft. 10 in., and the slant height, 2 ft. 2 in.

10. Find the volume of the frustum of a regular square pyramid whose height is 24 ft., and the sides of its square ends are respectively 9 ft. and 4 ft.

11. The slant height of the frustum of a regular square pyramid is 20 ft., the length of each side of its base 40 ft., of each side of its top 16 ft.; find its volume.

12. The mast of a ship is 51 ft. high, and the circumferences of its ends are 5 ft. 6 in. and 1 ft. 10 in. What will it cost at 60 cents per cubic foot?

13. A bucket is 16 in. deep, 18 in. wide at the top, and 12 in. wide at the bottom. How many gallons of water will it hold, reckoning $7\frac{1}{2}$ gals. to the cubic foot?

14. The frustum of a regular pyramid is 5 ft. high, and its bases are two regular octagons whose sides are 4 ft. and 3 ft. respectively. Find the volume of the frustum.

15. If the bases of the frustum of a pyramid are two regular hexagons whose sides are 1 ft. and 2 ft. respectively, and the volume of the frustum is 12 cu. ft., find its height.

16. A plane parallel to the base of a right cone cuts from the cone a frustum whose volume is 20^{cbm}, and the radii of the two bases are 3^m and 2^m respectively. Find the volume of the entire cone.

17. The radii of the bases of the frustum of a cone are 3.5^m and 7.3^m, and the height of the frustum is 2^m; find the surface and the volume of the entire cone.

18. The height of a right cone is 10^m, the radius of its base 5^m. What must be the distance from the base of a plane parallel to the base in order that the volume of the frustum made by the plane may be 20^{cbm}?

19. The frustum of a right cone is 14 ft. high, and has a volume of 924 cub. ft. Find the radii of its bases if their sum is 9 ft.

20. From a right cone whose slant height is 30 ft., and circumference of whose base is 10 ft., there is cut off by a plane parallel to the base a cone whose slant height is 6 ft. Find the convex surface and the volume of the frustum.

21. The interior dimensions of an iron vessel, open at the top and in the shape of the frustum of a right cone, are: diameter at the top 1 ft. 9 in., at the bottom 7 in., depth 1 ft. 8 in.; and the corresponding exterior dimensions are 2 ft., 9 in., and 1 ft. 10 in. How many cubic inches of iron were used in its construction?

22. A Dutch windmill in the shape of the frustum of a right cone is 12^m high. The outer diameters at the bottom and the top are 16^m and 12^m, the inner diameters 12^m and 10^m, respectively. How many cubic meters of stone were required to build it?

23. The chimney of a factory has the shape of the frustum of a regular pyramid. Its height is 180 ft., and its upper and lower bases are squares whose sides are 16 ft. and 10 ft., respectively. The section of the flue is throughout a square whose side is 7 ft. How many cubic feet of material does the chimney contain?

24. Find the volume V of the frustum of a cone of revolution, having given the slant height a, the height h, and the convex surface S.

§7. The Sphere.

1. What is the locus of the tangents which can be drawn from an exterior point to a given sphere?

2. The radius of a sphere is 2^m. Find the area of a section made by a plane, the distance of which from the centre of the sphere is 40^{cm}.

3. What is the distance of a small circle the area of which is 3^{qm}, from the centre of a sphere the radius of which is 2^m?

4. The radius of a sphere is 7 ft. Find the distance from the pole of a great circle to its circumference, (i.) when measured on the surface of the sphere; (ii.) when measured in a straight line.

5. The distance from the two poles of a small circle drawn on the surface of a sphere to the circumference of the circle are 3^m and 4^m respectively. Find the area of this circle.

6. In order to find the radius of a sphere, describe a small circle with a pair of dividers, and measure the length of its circumference with the help of a string. If the opening of the dividers is 6 in., and the length of the circumference is 22 in., find the radius of the sphere.

7. Find the surface of a sphere if the diameter is (i.) 10 in.; (ii.) 1 ft. 9 in.; (iii.) 2 ft. 4 in.; (iv.) 7 ft.; (v.) 4.2 ft.; (vi.) 10.5 ft.

8. Find the diameter of a sphere if the surface is (i.) 616 sq. in.; (ii.) 38½ sq. ft.; (iii.) 9856 sq. ft.

9. The circumference of a dome in the shape of a hemisphere is 66 ft.; how many square feet of lead are required to cover it?

10. How many square feet of lead are required to make 16 hemispherical bowls, if the diameter of each is 2 ft. 4 in.?

11. If the ball on the top of St. Paul's Cathedral in London is 6 ft. in diameter, what would it cost to gild it at 7 cents per square inch?

12. If the surface of a sphere is S, what is the surface of a sphere having a radius twice as large?

13. If the surface of a sphere is S, find the circumference C of a great circle.

14. What is the numerical value of the radius of a sphere if its surface has the same numerical value as the circumference of a great circle?

15. Find the surface of a lune if its angle is 30°, and the total surface of the sphere is 4 sq. ft.

16. What is the angle of a lune if its surface is 1 sq. ft., and the total surface of the sphere is 9 sq. ft.?

17. What fractional part of the whole surface of a sphere is a spherical triangle whose angles are 43° 27′, 81° 57′, and 114° 36′?

18. The angles of a spherical triangle are 60°, 70°, and 80°. The radius of the sphere is 14 ft. Find the area of the triangle in square feet.

19. The sides of a spherical triangle are 38°, 74°, and 128°. The radius of the sphere is 14 ft. Find the area of the polar triangle in square feet.

20. The sides of a spherical triangle are each equal to 90°. What part of the whole surface of the sphere does the triangle contain?

21. The radius of a sphere is 7 ft. Find the area of a spherical triangle formed by the arcs of three great circles, two of which are perpendicular to the third, and make with each other an angle of 60°.

22. Through a diameter of a sphere two planes are passed so as to cut from the circumference of a great circle, perpendicular to both the planes, arcs of 45°. The areas of the spherical surfaces included by these planes are 32π sq. ft. Find the radius of the sphere.

23. On a sphere whose radius is 21 ft., find the area of a spherical triangle polar to one whose perimeter is equal to half the circumference of a great circle.

24. Find the area of a spherical polygon on a sphere whose radius is $10\frac{1}{2}$ ft., if its angles are 100°, 120°, 140°, and 160°.

25. The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°; and the length of a lateral edge of the pyramid is 42 ft. Find the area of its base in square feet.

26. The planes of the faces of a triangular spherical pyramid make with each other angles of 40°, 60°, and 100°, and the area of the base of the pyramid is 4π sq. ft. Find the radius of the sphere.

27. The diameter of a sphere is 21 ft. Find the curved surface of a segment whose height is 5 ft.

28. What is the area of a zone of one base whose height is h, and the radius of the base r? What would be the area if the height were twice as great?

29. In a sphere whose radius is r, find the height of a zone whose area is equal to that of a great circle.

30. The radius of a sphere is 4 ft. The sphere is cut by two parallel planes situated on the same side of the centre, at the distances 2 ft. and 3 ft. respectively from the centre. Find the area of the zone formed by the planes, and the area of each of the circles which form its bases.

31. If the radius of a sphere is r, and the height of a zone on the sphere is h, find the radius of a circle equivalent to the area of the zone.

32. How many square feet of lead will be required to line the inside of a bowl in the shape of a segment of a sphere which measures 40 in. across the top, and whose greatest depth is 10 in.?

33. Find the convex surface of a slice 2 ft. high, cut from a sphere whose radius is 17 ft.

34. The height of a spherical zone is 8 ft., and the radii of its bases are 10 ft. and 6 ft. Find the area of the zone.

35. The altitude of the torrid zone is about 3200 miles. Find its area in square miles, assuming the earth to be a sphere with a radius of 4000 miles.

36. A plane divides the surface of a sphere of radius r into two zones, such that the surface of the greater is a mean proportional between the entire surface and the surface of the smaller. Find the distance of the plane from the centre of the sphere.

37. If a sphere of radius r is cut by two planes equally distant from the centre, so that the area of the zone comprised between the planes is equal to the sum of the areas of its bases, find the distance of either plane from the centre.

38. Find the area of the zone generated by an arc of 30°, of which the radius is r, and which turns around a diameter passing through one of its extremities.

39. Find the area of the zone of a sphere of radius r, illuminated by a lamp placed at the distance h from the sphere.

40. How much of the earth's surface would a man see if he were raised to the height of the radius above it?

41. To what height must a man be raised above the earth in order that he may see one-sixth of its surface?

42. Two cities are 200 miles apart. To what height must a man ascend from one city in order that he may see the other, supposing the circumference of the earth to be 25,000 miles?

43. Find the volume of a sphere if the diameter is (i.) 14 in.; (ii.) 3 ft. 6 in.; (iii.) 10 ft. 6 in.; (iv.) 17 ft. 6 in.; (v.) 14.7 ft.; (vi.) 42 ft.

44. Find the diameter of a sphere if the volume is (i.) 75 cub. ft. 1377 cub. in.; (ii.) 179 cub. ft. 1152 cub. in.; (iii.) 1047.816 cub. ft.; (iv.) 38.808 cub. yds.

45. Find the volume of a sphere whose circumference is 45 ft.

46. Find the volume V of a sphere in terms of the circumference C of a great circle.

47. Find the radius r of a sphere, having given the volume V.

48. Find the radius r of a sphere, if its circumference and its volume have the same numerical value.

49. Find the volume of a sphere if the section made by a plane 0.2^{m} from the centre contains 0.8^{qm}.

50. How many cubic inches of lead are required to make a spherical shell $\frac{1}{2}$ of an inch thick, if the exterior diameter is to be 7 in.?

51. How many gallons of water will a hemispherical bowl hold, whose diameter is 21 in., if 1 cub. ft. contains $7\frac{1}{2}$ gals.?

52. Find the weight of a solid ball of iron whose diameter is 5 in., if 1 cub. in. of iron weighs $4\frac{1}{2}$ oz.

53. What must be the exterior diameter of a spherical shell $\frac{1}{2}$ of an inch thick, that its hollow part may contain $113\frac{1}{7}$ cub. in.?

54. If the earth's diameter be 8000 miles, and geologists knew the interior to the depth of 5 miles below the surface, what fraction of the whole volume would be known?

55. Find the diameter of the mouth of a cannon that carries a ball weighing 24 lbs., when 1 cub. in. of iron weighs $4\frac{1}{2}$ oz.

56. What is the weight of a spherical shell 10 in. in diameter and 2 in. thick, composed of a substance of which 1 cub. ft. weighs 216 lbs.?

57. The weight of an iron ball whose diameter is 4 in. is 9 lbs.; find the weight of a shell whose external and internal diameters are 8 in. and 5 in. respectively.

58. If an iron ball 4 in. in diameter weighs 9 lbs., what is the weight of a hollow iron shell 2 in. thick, whose external diameter is 20 in.?

59. The radius of a sphere is 7 ft.; what is the volume of a wedge whose angle is 36°?

60. What is the angle of a spherical wedge, if its volume is 1 cub. ft., and the volume of the entire sphere is 6 cub. ft.?

61. What is the volume of a spherical sector, if the area of the zone which forms its base is 3 sq. ft., and the radius of the sphere is 1 ft.?

62. The volume of a spherical sector is 0.48^{cbm}, and the radius of the sphere is 2^{m}; find the area of the base of the sector?

63. If the volume of a spherical sector is a, and the area of its base b, what is the volume V of the sphere to which the sector belongs?

64. Find the volume of a triangular spherical pyramid, if the angles of the spherical triangle which forms its base are each 100°, and the radius of the sphere is 7 ft.?

65. The circumference of a sphere is 28π ft.; find the volume of that part of the sphere included by the faces of a trihedral angle at the centre, the face angles of which are 80°, 105°, and 140°.

66. The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°, and a lateral edge of the pyramid is $3\frac{1}{2}$ ft.; find the volume of the pyramid.

67. The radius of the base of the segment of a sphere is 40 ft., and its height is 20 ft.; find its volume.

68. The radius of the base of the segment of a sphere is 16 in., and the radius of the sphere is 20 in.; find its volume.

69. The inside of a wash-basin is in the shape of the segment of a sphere; the distance across the top is 16 in., and its greatest depth is 6 in.; find how many pints of water it will hold, reckoning $7\frac{1}{2}$ gallons to the cubic foot.

70. What is the height of a zone, if its area is S, and the volume of the sphere to which it belongs is V?

71. The radii of the bases of a spherical segment are 6 ft. and 8 ft., and its height is 3 ft.; find its volume.

72. Find the volume of a spherical segment, if the diameter of each base is 8 ft., and the height is 4 ft.

73. A bowl has the shape of a spherical segment of two bases. It measures 2 ft. across the top, and 10 in. across the bottom, and its greatest depth is 13 in.; how many gallons of water will it hold, if 1 cub. ft. contains $7\frac{1}{2}$ gallons?

74. A sphere whose radius is 7 ft. is cut by two parallel planes on the same side of the centre, at the distances from the centre of 3 ft. and 5 ft. respectively; find the volume of the spherical segment comprised between the planes.

75. The radius of a sphere is 10^m; find the volume of the segment cut from the sphere by a plane whose distance from the centre is equal to half the radius.

76. A sphere is cut by a plane whose distance from the centre is equal to half the radius; find the ratio of the two parts into which this plane divides (i.) the surface of the sphere; (ii.) the volume of the sphere.

77. A spherical segment of one base is half as large as the spherical sector to which it belongs; having given the radius r of the sphere, find the height of the segment.

78. The radii of two concentric spheres are r, r'; find the volume of the portion contained between the two surfaces and two parallel planes passing on the same side of the common centre at the distances a and $a + b$ from the centre.

79. Having given the volume V, and the height h, of a spherical segment of one base, find the radius r of the sphere.

80. Find the weight of a sphere of radius r, which floats in a liquid of specific gravity s, with $\frac{1}{4}$ of its surface above the surface of the liquid. *The weight of a floating body is equal to the weight of the liquid displaced.*

§ 8. Equivalent Solids.

1. What is the ratio of the heights of two equivalent right prisms, if the areas of their bases are as 4 : 5?

2. A cube whose edge is 12^{cm} long is transformed into a right prism whose base is a rectangle 16^{cm} long and 12^{cm} wide; find the height of the prism, and the difference between its total surface and the surface of the cube.

3. The dimensions of a rectangular parallelopiped are a, b, c. Find (i.) the height of an equivalent right cylinder having a for the radius of its base; (ii.) the height of an equivalent right cone having a for the radius of its base; (iii.) the radius of an equivalent sphere.

4. A cube of lead whose edge is a is melted and recast in the shape of a sphere; find the radius of the sphere.

5. A sphere of lead whose radius is r is melted and recast in the shape of a cube; find the length of one edge of the cube.

6. A regular pyramid 12 ft. high is transformed into a regular prism with an equivalent base; what is the height of the prism?

7. The diameter of a cylinder is 14 ft., and its height is 8 ft.; find the height of an equivalent right prism, the base of which is a square with a side 4 ft. long.

8. If one edge of a cube is a, what is the height h of an equivalent right cylinder whose diameter is b?

9. The diameters of two equivalent cylinders are as 1 : 4. The height of the first is 48 ft.; what is the height of the other?

10. The heights of two equivalent right cylinders are as 4 : 9. The diameter of the first is 6 ft.; what is the diameter of the other?

11. A right cylinder 6 ft. in diameter is equivalent to a right cone 7 ft. in diameter. If the height of the cone is 8 ft., what is the height of the cylinder?

12. A sphere whose diameter is d is transformed into an equivalent cylinder having the same diameter; find the height h of the cylinder.

13. A bullet 3 in. in diameter is recast in the form of a cylinder 2 in. in diameter; find the height of this cylinder.

14. Find the height of a right cone equivalent to a hemisphere 8 ft. in diameter, and having the same base as that of the hemisphere.

15. A sphere and an equivalent right cylinder have equal diameters; compare their total surfaces.

16. The frustum of a regular four-sided pyramid is 6 ft. high, and the sides of its bases are 5 ft. and 8 ft. respectively. What is the height of an equivalent regular pyramid whose base is a square with a side 12 ft. long?

17. The frustum of a cone of revolution is 5 ft. high, and the diameters of its bases are 2 ft. and 3 ft., respectively; find the height of an equivalent right cylinder whose base is equal in area to the section of the frustum made by a plane parallel to its bases, and equidistant from the bases.

18. The radii of two spheres are r, r'. Find the radius of a sphere equivalent to both the spheres.

19. 100 bullets, equal in size, are thrown into a cylindrical vessel 8 in. in diameter containing water. If the level of the water rises $4\frac{1}{6}$ in., find the diameter of a bullet.

§ 9. Similar Solids.

1. The edges of two cubes are 6 in. and 7 in.; what is the ratio of their surfaces? their volumes?

2. If the edge of a cube is a, what is the edge of a cube twice as large?

3. The dimensions of a trunk are 4 ft., 3 ft., 2 ft. What are the dimensions of a trunk similar in shape that will hold 4 times as much?

4. The diameters of two spheres are 5 in. and 20 in., respectively. What is the ratio of their surfaces? their volumes?

5. The radii of the earth, the moon, and the sun are as $1 : \frac{3}{11} : 112$. If the volume of the earth is taken as unity, what will be the volumes of the moon and the sun?

6. The radius of a sphere is 1 ft. What is the radius of a sphere 5 times as large?

7. The radius of a sphere is 1 ft. What is the radius of a sphere whose surface is double that of the given sphere?

8. A sphere 9 in. in diameter is reduced in size by a lathe until the diameter is 8 in. What part of the whole volume is removed?

9. The diameter of an iron ball is 6 in. What is the diameter of an iron ball weighing $\frac{1}{27}$ as much?

10. Compare the surfaces and the volumes of two spheres, if the diameter of one is $\frac{2}{3}$ that of the other.

11. If a leaden ball 1 in. in diameter weighs $\frac{3}{14}$ of a pound, what is the diameter of a leaden ball that weighs 588 pounds?

12. The diameter of a globe is 7 in. Find the diameter of a globe whose volume is 3 times that of the first.

13. The diameter of a sphere is 1 ft. 9 in. Find the diameter of another sphere whose volume is $\frac{1}{8}$ that of the former.

14. How much must the dimensions of a cylinder be increased in order to obtain a similar cylinder (i.) whose surface shall be n times that of the first; (ii.) whose volume shall be n times that of the first?

15. A pyramid is cut by a plane which passes midway between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off.

16. The height of a regular hexagonal pyramid is 36 ft., and one side of the base is 6 ft. What are the dimensions of a similar pyramid whose volume is $\frac{1}{20}$ that of the first?

17. The length of one of the lateral edges of a pyramid is 4^m. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?

18. The length of a lateral edge of a pyramid is a. At what distances from the vertex will this edge be cut by two planes parallel to the base, which divide the pyramid into three equivalent parts?

19. The length of a lateral edge of a pyramid is a. At what distance from the vertex will this edge be cut by a plane parallel to the base, and dividing the pyramid into two parts which are to each other as $3 : 4$?

20. The length of a lateral edge of a pyramid is a. At what distance from the vertex must this edge be cut by a plane parallel to the base, in order that the volume of the pyramid cut off may be $\frac{1}{8}$ of the volume of the frustum?

21. The volumes of two similar cones are 54 cub. ft. and 432 cub. ft. The height of the first is 6 ft.; what is the height of the other?

22. The height of a cone is h. At what distance from the base must a plane parallel to the base be passed, in order that the volume of the cone cut off by the plane may be $\frac{1}{7}$ of the volume of the frustum which remains?

23. The slant height of a cone of revolution is a. It is required to divide the cone by planes parallel to the base into three parts, the volumes of which shall be proportional to the numbers 2, 3, and 5. At what distances from the vertex must the planes be passed?

24. The height of a cone of revolution is h. At what distances from the vertex must planes parallel to the base be passed, in order that they may divide the convex surface of the cone into three equivalent parts?

25. The height of a cone of revolution is h, and the radius of its base is r. What are the dimensions of a similar cone three times as large?

26. The height of the frustum of a right cone is $\frac{2}{3}$ the height of the entire cone. Compare the volumes of the frustum and the entire cone.

27. The frustum of a pyramid is 8 ft. high, and two homologous edges of its bases are 4 ft. and 3 ft., respectively. Compare the volume of the frustum and that of the entire pyramid.

28. In each of two right cylinders the diameter is equal to the height. The volume of one is $\frac{3}{4}$ that of the other. What is the ratio of their heights?

29. Find the dimensions of a right cylinder $\frac{5}{16}$ as large as a similar cylinder whose height is 20 ft., and diameter 10 ft.

30. If a box will hold 1200 balls 1 in. in diameter, how many balls 2 in. in diameter will it hold?

31. The interior diameter of a hollow hemispherical dome is 20 ft., and the thickness of the dome at all points is 1 ft. If it costs \$5.00 to paint the interior surface, what will it cost at the same rate per square foot to paint the exterior surface?

32. The exterior diameter of a spherical shell is 20^{cm}, and its weight is $\frac{1}{10}$ that of a solid ball made of the same material and having the same diameter. Find the thickness of the shell.

§ 10. Solids of Revolution.

1. The sides of a rectangle are as 2 : 1. What is the ratio of the volumes of the solids generated by the revolution of the rectangle about these sides, respectively, as axes?

2. The sides of a right triangle are 3, 4, 5. Compare the volumes of the solids generated by revolving the triangle about these three sides respectively.

3. Compare the volumes of the solids generated by revolving an isosceles right triangle about one of its legs and about its hypotenuse.

4. Compare the areas of the curved surfaces generated by revolving an isosceles right triangle about one of its legs and about its hypotenuse.

5. Having given the legs a, b of a right triangle, find the volume of the solid generated by revolving the triangle about its hypotenuse.

6. Having given the legs a, b of a right triangle, find the area of the surface generated by these two legs when the triangle revolves about its hypotenuse.

7. An equilateral triangle, whose side is a, turns about one of its sides; find the surface and the volume of the solid which is generated.

8. An equilateral triangle, whose side is a, revolves about an axis parallel to one side and passing through the opposite vertex. Find the volume of the solid generated.

9. The base of an isosceles triangle = 6 ft., and one of the equal sides = 9 ft. If the triangle turn about an axis passing through the vertex and parallel to the base, find the volume of the solid which is generated.

10. Find the area of the curved surface generated by a line AB, 5^m long, turning about an axis in the same plane, the distances of A, B from the axis being 3^m and 4^m, respectively.

11. Find the volume of the solid generated by an equilateral triangle, whose side = a, turning about an axis perpendicular to one of its sides, and passing through one end of this side.

12. One side of an equilateral triangle is produced by its own length, and at the extremity a perpendicular is erected. If the length of the side is a, find the volume of the solid formed by revolving the triangle about this perpendicular as an axis.

13. The side of an equilateral triangle = a. The triangle is revolved about an axis parallel to one side and at the distance b from this side. Find the surface and the volume of the solid which is generated.

14. Having given the hypotenuse a of a right triangle, find the two legs, if the volumes of the two cones, generated by revolving the triangle about the legs, are as 2 : 1.

15. Find the volume of the solid generated by a triangle whose sides are 2^m, 3^m, and 4^m, and which revolves about its longest side.

16. The radius of a circle = 2 ft. Through a point A, 3 ft. from the centre of the circle, two tangents are drawn, touching the circle in B, C, respectively. Find the volume of the solid generated by revolving the triangle ABC about the diameter parallel to BC.

17. At the middle point M of the radius OA of a circle, a perpendicular is erected meeting the circumference in N. Through N a tangent is drawn meeting OA produced in P. Find the volume of the solid generated by the triangle MNP turning about OP as an axis, the radius of the circle being r.

18. Find the surface and the volume generated by a square whose side is a, revolving about an axis passing through one end of a diagonal of the square, and perpendicular to this diagonal.

19. A rectangle $ABCD$ revolves about an axis parallel to AB and situated without the rectangle. Prove that the volume of the solid generated is equal to the area of the rectangle multiplied by the circumference described by the intersection of its two diagonals.

20. Compare the volumes of the solids generated by a parallelogram turning successively about two adjacent sides, the lengths of these sides being a and b.

21. A regular hexagon, whose side is a, turns about one of its sides. Find the surface and the volume of the solid which is generated.

22. A circle is circumscribed about a regular hexagon whose side is a, and then revolved about a diameter passing through two opposite vertices of the hexagon. Compare the volumes of the solids generated by the circle and the hexagon.

23. A circle is circumscribed about a regular hexagon whose side is a, and then revolved about a diameter passing through the middle points of two opposite sides of the hexagon. Compare the volumes of the solids generated by the circle and the hexagon.

24. What must be the angle of a circular sector AOC, in order that the volume of the spherical sector, generated by the revolution of the circular sector about AO, may be exactly $\frac{1}{4}$ of the entire volume of the sphere to which it belongs?

25. The diameter of a circle is 4 ft. Find the surface and the volume of the solid generated by a chord 2 ft. long, revolving about a diameter parallel to the chord.

26. Upon the diameter AOB of a semicircle, of which O is the centre and r the radius, semicircles are described having for their diameters AO and OB. The figure is then revolved about AOB as an axis. Find the volume of the solid generated by the surface comprised between the three semicircles.

27. An equilateral triangle revolves about one of its altitudes. Compare the convex surface of the cone generated by the triangle and the surface of the sphere generated by the circle inscribed in the triangle.

28. An equilateral triangle revolves about one of its altitudes. Compare the volumes of the solids generated by the triangle, the inscribed circle, and the circumscribed circle.

29. A circle is inscribed in a square, and the figure is revolved about the diameter, passing through two opposite points of contact. Compare the convex surface of the cylinder generated by the square, and the surface of the sphere generated by the circle.

30. A circle is inscribed in a square, and one of the points of contact is joined to the extremities of the opposite side of the square. The figure is then revolved about the diameter which passes through this point of contact. Compare the volumes of the cylinder, sphere, and cone which are generated.

§ 11. Inscribed and Circumscribed Solids.

1. Find the volume in cubic feet of the largest square beam which can be made from a log in the shape of a right cylinder 40 ft. long and 2 ft. in diameter.

2. The edge of a cube is 14 ft. What is the convex surface of the largest cylinder that can be made from it?

3. Compare the volumes of the largest cylinder, sphere, and cone which can be made from a given cube.

4. Compare the volumes of the largest cylinder, sphere, and cone which can be made from a right prism a ft. long, the base of which is a square with a side b ft. long.

5. What is the ratio of the volume of a regular quadrangular pyramid to that of the inscribed cone?

6. What is the ratio of the volume of a cone of revolution to that of the inscribed regular quadrangular pyramid?

7. If a wooden cube weighs 21 lbs., what will be the weight of the largest cone which can be turned from it?

8. The edge of a regular tetrahedron is a. Find the radii r, r' of the inscribed and circumscribed spheres, the surface S, and the volume V.

9. The same exercise for the regular octahedron.

10. Compare the volume of the cube inscribed in a given sphere with that of the cube circumscribed about the same sphere.

11. The same exercise for the regular tetrahedron.

12. The same exercise for the regular octahedron.

13. The height of a frustum of a right cone is h, and the radii of its bases r, r'; what is the volume of the largest regular four-sided frustum which can be made from it?

14. The volume of a right cone, whose slant height is equal to the diameter of its base, is V; find the volume of the circumscribed regular triangular pyramid.

15. A cone, whose slant height is equal to the diameter of its base, is inscribed in a given sphere, and a similar cone is circumscribed about the same sphere. Compare the volumes of these two cones.

16. The radius of the base of a right cone is a, and the radius of the inscribed sphere is b; find the volume of the cone.

17. The volume of the frustum of a right cone is V; find the difference between the volumes of the circumscribed and the inscribed regular hexagonal frustums.

18. The volume of the frustum of a regular quandrangular pyramid is V; find the difference between the volumes of the circumscribed and the inscribed frustums of cones.

19. The height of a cone circumscribed about a sphere is double the diameter of the sphere. If the surface of the sphere is S, find the total surface of the cone.

20. Find the height of a right cone inscribed in a sphere of radius r, if the convex surface of the cone is equal to twice the area of its base.

USEFUL FORMULÆ.

V stands for volume. S stands for the lateral surface of the *right* prism, *regular* pyramid, or frustum of *regular* pyramid, and the convex surface of the cylinder of revolution, cone of revolution, or frustum of cone of revolution; T stands for the total surface of each of these solids, respectively.

The Prism.

$S =$ perimeter of base $\times$ height.
$T = S +$ twice area of base.
$V =$ area of base $\times$ height.

The Cylinder of Revolution.

Radius of base, r; height, h.

$S = 2\pi rh.$
$T = 2\pi rh + 2\pi r^2.$
$V = \pi r^2 h.$

The Pyramid.

$S = \frac{1}{2}$ perimeter of base $\times$ slant height.
$T = S +$ area of base.
$V = \frac{1}{3}$ area of base $\times$ height.

The Cone of Revolution.

Radius of base, r; height, h; slant height, a.

$S = \pi ra.$
$T = \pi ra + \pi r^2.$
$V = \frac{1}{3}\pi r^2 h.$

Frustum of Pyramid.

Perimeters of bases, p and p'; areas of bases, B and B'; height, h.

$S = \frac{1}{2}(p + p') \times$ slant height.
$T = S + B + B'.$
$V = \frac{1}{3}h(B + B' + \sqrt{BB'}).$

Frustum of Cone of Revolution.

Radii of bases, r and r'; height, h; slant height, a.

$$S = \pi a (r + r').$$
$$T = \pi a (r + r') + \pi (r^2 + r'^2).$$
$$V = \tfrac{1}{3} \pi h (r^2 + r'^2 + rr').$$

The Sphere.

Radius, r; angle of lune, α; angles of spherical Δ, α, β, γ; height of zone or segment, h.

Surface of sphere $= 4\pi r^2$.

Area of lune $= \dfrac{\pi r^2 \alpha}{90}$.

Area of spherical $\Delta = \dfrac{(\alpha + \beta + \gamma - 180°)}{720°} \times 4\pi r^2$.

Area of zone $= 2\pi rh$.

Vol. of sphere $= \tfrac{4}{3} \pi r^3$.

Vol. of wedge $= \tfrac{1}{3} r \times$ area of base.

Vol. of sph. pyramid $= \tfrac{1}{3} r \times$ area of base.

Vol. of sph. sector $= \tfrac{2}{3} \pi r^2 h$.

Vol. of sph. segment $= \tfrac{1}{2} h$ (sum of bases) $+ \tfrac{1}{6} \pi h^3$.

USEFUL NUMBERS.

$\sqrt{2} = 1.41421.$	$\sqrt[3]{2} = 1.25992.$
$\sqrt{3} = 1.73205.$	$\sqrt[3]{3} = 1.44224.$
$\sqrt{5} = 2.23606.$	$\sqrt[3]{5} = 1.70998.$
$\sqrt{7} = 2.64575.$	$\sqrt[3]{7} = 1.91293.$

ANSWERS

TO

EXERCISES IN SOLID GEOMETRY.

§ 1. Planes.

10. 12 in.
11. 10 in.
12. 7.071 in.
13. 12 in.
14. 20.44 in.
15. 24°, 156°.
17. 5.143 in., 6.857 in.
33. 2.828 in.
34. 60°.

§ 2. The Prism.

1. $6a^2$; a^3; $a\sqrt{3}$.

2. 4 ft.; 64 cub. ft.
$\sqrt{\frac{S}{6}}$; $\frac{1}{6}\sqrt{\frac{S^3}{6}}$.

3. 10 ft.; 600 sq. ft.
$\sqrt[3]{V}$; $6\sqrt[3]{V^2}$.

4. 20.

5. Each edge = 1^m.

6. $\frac{a^3\sqrt{2}}{4}$.

7. 405 sq. ft.

8. 512 cub. ft.

9. 4 ft.

10. 2250 lbs.

11. 82.752^{kg}.

12. 62 sq. ft. 24 sq. in.

13. 15,360 bricks.

14. 1 ft. 8 in.

15. 4145 lbs. 13 oz.; 6255 lbs. 3 oz.

16. 1 cub. ft. 270 cub. in.

17. 1815 cub. ft.

18. $2(ab+ac+bc)$; abc; $\sqrt{a^2+b^2+c^2}$.

19. $\sqrt{\frac{m^2V}{np}}$, $\sqrt[3]{\frac{n^2V}{mp}}$, $\sqrt[3]{\frac{p^2V}{mn}}$.

20. $S=3ah$; $V=\frac{a^2h\sqrt{3}}{4}$.

21. $S=4ah$; $V=a^2h$.

22. $S=6ah$; $V=\frac{3a^2h\sqrt{3}}{2}$.

23. $S=8ah$; $V=2a^2h(1+\sqrt{2})$.

24. 168 cub. in.

25. 31 cub. ft. 864 cub. in.

26. 792 sq. in.; 1440 cub. in.

27. 18 cub. ft. 73.2 cub. in.

28. 1920 cub. ft.

29. 6^{cbm}.

30. 187.0632 cub. in.

31. 140,000 cub. ft.

32. 117,333,333⅓ cub. yds.

33. 1558.8 cub. ft.

34. $a=\sqrt{\frac{2V}{3h\sqrt{3}}}$.

§ 3. The Cylinder of Revolution.

1. 251.4 sq. ft.
2. $S = 73$ sq. ft. 48 sq. in.; $V = 42$ cub. ft. 1344 cub. in.
3. 7 in.
4. 20 ft.
5. 12.53 in.
6. $h = \frac{a}{2\pi r} - r.$
7. $r = \frac{2b}{a}, \quad h = \frac{a^2}{4\pi b}.$
8. 4, 2, and 8 times, respectively.
9. $31.43.
10. $1746\frac{2}{3}$ cub. yds.
11. $2887\frac{1}{2}$ gals.
12. 7392 yds.
13. 294 lbs. $10\frac{4}{7}$ oz.
14. 154 sq. in.; 2156 cub. in.
15. 170.903^{kg}.
16. $r = 1.469^{dm}$; $h = 2.938^{dm}$.
17. 2.26^{qm}.
18. 1.768^{kg}.
19. 1 cub. ft. $1157\frac{1}{7}$ cub. in.
20. $165; $12.83.
21. 1 ft. $3\frac{5}{7}$ in.
22. 16 ft. 6.88 in.
23. 1540 revolutions.
24. $2346\frac{2}{3}$ ft.
25. 95 lbs. 12 oz.
26. 9249^{l}.
27. 84823.2^{ccm}.
28. 546.89^{kg}.
29. $V = \frac{c^2 h}{4\pi}.$
30. $V = \frac{T}{3}\sqrt{\frac{T}{6\pi}}.$
31. $V = \frac{c}{8\pi^2}(2\pi T - c^2).$
32. $T = 2\left(\frac{V}{h} + \sqrt{\pi V h}\right).$
33. $h = \sqrt[3]{\frac{4V}{\pi}}, \quad T = 3\sqrt[3]{2\pi V^2}.$
34. $d = -h + \sqrt{\frac{2T}{\pi} + h^2};$

$$V = \frac{h}{2}\{\pi h^2 + T - h\sqrt{\pi(2T + \pi h^2)}\}.$$

§ 4. The Pyramid.

1. 105 sq. ft.
2. 11.54 ft.
3. 6 ft.
4. 7.11 sq. ft.
5. 14.87 sq. ft.; 23.5 sq. ft.
6. 25.85 ft.
7. $33\frac{1}{3}$ cub. ft.
8. 34.64 cub. ft.
9. 935.306 cub. ft.
10. 384 sq. ft.

11. 177.588 sq. ft.
12. 5096 sq. ft.
13. $6\frac{1}{2}$ ft.
14. 10 ft.
15. 51,710.933 cub. ft.
16. 266.29 sq. ft.
17. Slant height, 12^{cm}; area of one face, 108^{qcm}; height, 10.816^{cm}; total surface, 464.29^{qcm}; volume, 505.85^{cbm}.
18. 185 sq. ft.
19. 198 oz. Avoir.
20. 475.45 cub. ft.
21. $V=\frac{a}{6}\sqrt{T(T-2a^2)}$.
22. $h=\frac{1}{2a}\sqrt{T(T-2a^2)}$.
23. $\sqrt{\frac{T}{1+\sqrt{3}}}$.
24. $\frac{T}{2}\sqrt{\frac{2}{T+2h^2}}$.

§ 5. The Cone of Revolution.

1. 0.16^{qm}.
2. 6 ft.
3. 2.309 ft.
4. Area of section $=\frac{\pi bc^2}{\pi a^2-b}$.
5. 1.414 ft.
6. $\frac{1}{3}\sqrt{5}$.
7. 12 ft.
8. 28 ft.
9. 3 cub. ft. 1116 cub. in.
10. $351.
11. 96.052 sq. ft.
12. $38\frac{1}{2}$ sq. ft.
13. $67.13.
14. $27\frac{59}{63}$ yds.
15. $\frac{3}{4}\pi a^2$.
16. $2V$; $4V$; $8V$.
17. 103,488 cub. yds.
18. 18.935 ft.
19. $77\frac{21}{22}$ cub. ft.
20. 34 cub. ft. $310\frac{1}{2}$ cub. in.
21. $116\frac{27}{154}$.
22. $2256\frac{48}{49}$ gals.
23. $3\frac{1}{7}$ cub. in.
24. $5\frac{2}{15}$ cub. ft.
25. $r=3.85^{cm}$; $h=15.4^{cm}$.
26. $21\frac{1}{3}$ cub. ft.
27. 3.2969^{qm}.
28. 4.7952^{cm}.
29. 1982.4^{ccm}.
30. $\frac{T}{9}\sqrt{\frac{T}{\pi}}$.
31. $\frac{r}{3}\sqrt{T(T-2\pi r^2)}$.
32. $\frac{1}{3}\sqrt{\frac{T(T-S)(2S-T)}{\pi}}$.

§ 6. Frustums of Pyramids and Cones.

1. 11 sq. ft. 66 sq. in.
2. 430.3924 sq. ft.
3. $132.
4. $40\pi = 125.6636^{qm}$.
5. 5.28^{m}.
6. 34.22 sq. ft.
7. $\sqrt{\frac{a^2+ar}{2}}$.
8. 123.41 cub. ft.
9. $73\frac{20}{27}$ cub. ft.
10. 1064 cub. ft.
11. 13,312 cub. ft.
12. $35.45.
13. $12\frac{17}{84}$ gals.
14. 297.75 cub. ft.
15. 1.98 ft.
16. 28.421^{cbm}.
17. 356.2^{qm}; 214.282^{cbm}.
18. 0.2816^{m}.
19. 6 ft. and 3 ft.
20. 144 sq. ft.; 78.83 cub. ft.
21. $1693\frac{10}{21}$ cub. in.
22. 716.2848^{cbm}.
23. 22,140 cub. ft.
24. $V = \frac{1}{12}\pi h\left(\frac{3S^2}{\pi^2 a^2}+a^2-h^2\right)$.

§ 7. The Sphere.

2. 12.06^{qm}.
3. 1.74^{m}.
4. 11 ft.; 9.898 ft.
5. 18.09^{qm}.
6. 3.69 in.
7. (i.) 2 sq. ft. $26\frac{2}{7}$ sq. in.;
   (ii.) 9 sq. ft. 90 sq. in.;
   (iii.) 17 sq. ft. 16 sq. in.;
   (iv.) 154 sq. ft.;
   (v.) 55.44 sq. ft.;
   (vi.) 346.5 sq. ft.
8. (i.) 1 ft. 2 in.;
   (ii.) 3 ft. 6 in.;
   (iii.) 56 ft.
9. 693 sq. ft.
10. $136\frac{8}{9}$ sq. ft.
11. $1140.48.
12. $4S$.
13. $C = \sqrt{\pi S}$.
14. $\frac{1}{2}$.
15. $\frac{1}{3}$ sq. ft.
16. 40°.
17. $\frac{1}{12}$.
18. $102\frac{2}{3}$ sq. ft.
19. $410\frac{2}{3}$ sq. ft.
20. $\frac{1}{8}$.
21. $51\frac{1}{8}$ sq. ft.
22. 8 ft.
23. 1386 sq. ft.
24. 308 sq. ft.
25. 2772 sq. ft.
26. 6 ft.

27. 330 sq. ft.

28. $\pi(h^2+r^2)$; $2\pi(h^2+r^2)$.

29. $h=\frac{r}{2}$.

30. Area of zone, $25\frac{1}{7}$ sq. ft.; areas of bases, 22 sq. ft. and $37\frac{5}{7}$ sq. ft.

31. Radius of circle $=\sqrt{2rh}$.

32. 10 sq. ft. $131\frac{3}{7}$ sq. in.

33. 213.714 sq. ft.

34. $502\frac{6}{7}$ sq. ft.

35. 80,457,143 sq. mi., nearly.

36. $r(\sqrt{5}-2)$.

37. $r(\sqrt{2}-1)$.

38. $\pi r^2(2-\sqrt{3})$.

39. $\frac{2\pi r^2 h}{r+h}$.

40. $\frac{1}{4}$.

41. $\frac{1}{2}$ the radius.

42. 5.03167 miles.

43. (i.) $1437\frac{1}{3}$ cub. in.;
(ii.) $22\frac{11}{24}$ cub. ft.;
(iii.) $606\frac{3}{8}$ cub. ft.;
(iv.) $2807\frac{7}{24}$ cub. ft.;
(v.) 1663.893 cub. ft.;
(vi.) 38,808 cub. ft.

44. (i.) 5 ft. 3 in.; (iii.) 12.6 ft.; (ii.) 7 ft.; (iv.) 4.2 yds.

45. 1538.8 cub. ft.

46. $V=\frac{C^3}{6\pi^2}$.

47. $r=\sqrt[3]{\frac{3V}{4\pi}}$.

48. $r=\frac{1}{2}\sqrt{6}=1.2247$.

49. 0.665^{cbm}.

50. $66\frac{11}{21}$ cub. in.

51. $10\frac{85}{256}$ gals.

52. 18 lbs. $6\frac{9}{14}$ oz.

53. 7 in.

54. $\frac{8}{800}$, nearly.

55. 5.46 in.

56. $51\frac{1}{8}$ lbs.

57. $54\frac{27}{44}$ lbs.

58. 549 lbs.

59. $143\frac{11}{16}$ cub. ft.

60. 60°.

61. 1 cub. ft.

62. 0.72^{qm}.

63. $V=\frac{36\pi a^3}{b^3}$.

64. $239\frac{4}{5}$ cub. ft.

65. $179\frac{3}{8}$ cub. ft.

66. $22\frac{11}{24}$ cub. ft.

67. $54{,}476\frac{4}{21}$ cub. ft.

68. 2 cub. ft. $30\frac{10}{21}$ cub. in.

69. $24\frac{37}{42}$ pts.

70. $h=\frac{S}{\sqrt[3]{6\pi^2 V}}$.

71. $485\frac{4}{7}$ cub. ft.

72. $234\frac{3}{8}$ cub. ft.

73. $19\frac{59}{60}\frac{23}{48}$ gals.

74. $205\frac{1}{3}$ cub. ft.

75. 654.498^{cbm}.

76. 1 : 3; 5 : 27.

77. $h=\frac{1}{2}r(3+\sqrt{5})$.

78. $V=\pi b(r^2-r'^2)$.

79. $r=\frac{V}{\pi h^2}+\frac{h}{3}$.

80. $\frac{2}{5}\pi r^3 s$.

§ 8. Equivalent Solids.

1. 5 : 4.
2. 9^{qcm}; 24^{qcm}.
3. (i.) $\frac{bc}{\pi a}$; (ii.) $\frac{3bc}{\pi a}$; (iii.) $\sqrt[3]{\frac{3abc}{4\pi}}$.
4. $\sqrt[3]{\frac{3a^3}{4\pi}}$.
5. $\sqrt[3]{\frac{4\pi r^3}{3}}$.
6. 4 ft.
7. 77 ft.
8. $h = \frac{4a^3}{\pi b^2}$
9. 3 ft.
10. 4 ft.
11. 3.629 ft.
12. $h = \frac{2}{3}d$.
13. $4\frac{1}{2}$ in.
14. 8 ft.
15. 6 : 7.
16. $5\frac{2}{3}$ ft.
17. 5.0666 ft.
18. $\sqrt[3]{r^3 + r'^3}$.
19. 1.5869 in.

§ 9. Similar Solids.

1. 36 : 49; 216 : 343.
2. $a\sqrt[3]{2}$.
3. 6.35 ft.; 4.76 ft.; 3.17 ft.
4. 1 : 16; 1 : 64.
5. $\frac{27}{1331}$; 1,404,928.
6. $\sqrt[3]{5}$.
7. 1.414 ft.
8. $\frac{217}{729}$.
9. 2 in.
10. Surfaces are as 25 : 64; volumes are as 125 : 512.
11. 14 in.
12. 10.094 in.
13. $10\frac{1}{2}$ in.
14. (i.) $\sqrt{n}$ times; (ii.) $\sqrt[3]{n}$ times.
15. 8 : 1.
16. Height, 13.2625 ft.; side of base, 2.2104 ft.
17. 3.17m.
18. $a\sqrt[3]{\frac{1}{3}}$, and $a\sqrt[3]{\frac{2}{3}}$.
19. $a\sqrt[3]{\frac{2}{5}}$.
20. $a\sqrt[3]{\frac{7}{5}}$.
21. 12 ft.
22. $\frac{h}{2}$.
23. $a\sqrt[3]{\frac{1}{5}}$, and $a\sqrt[3]{\frac{1}{2}}$.
24. $\frac{h\sqrt{3}}{3}$, and $\frac{h\sqrt{6}}{3}$.
25. Height $= h\sqrt[3]{3}$; radius of base $= r\sqrt[3]{3}$.
26. 98 : 125. 27. 37 : 64.

28. 1 : 0.90856.
29. Height, 19.5743; diameter, 9.7871.
30. 150 balls.
31. $6.05.
32. 0.3451^{cm}.

§ 10. Solids of Revolution.

1. 1 : 2.
2. 20 : 15 : 12.
3. $2 : \sqrt{2}$.
4. 1 : 1.
5. $V = \frac{\pi a^2 b^2}{3\sqrt{a^2+b^2}}$.
6. $S = \frac{\pi ab(a+b)}{\sqrt{a^2+b^2}}$.
7. $S = \pi a^2\sqrt{3}$; $V = \frac{\pi a^3}{4}$.
8. $V = \frac{\pi a^3}{2}$.
9. $V = 216\pi$ cub. ft.
10. $S = 109.95^{qm}$.
11. $V = \frac{\pi a^3\sqrt{3}}{4}$.
12. $V = \frac{3\pi a^3\sqrt{3}}{4}$.
13. $S = \pi a(a\sqrt{3}+6b)$; $V = \frac{1}{4}\pi a^2(a+2b\sqrt{3})$.
14. $\frac{a\sqrt{5}}{5}$ and $\frac{2a\sqrt{5}}{5}$.
15. 8.838^{cbm}.
16. $\frac{340\pi\sqrt{5}}{81}$ cub. ft.
17. $V = \frac{3\pi r^3}{8}$.
18. $S = 4\pi a^2\sqrt{2}$; $V = \pi a^3\sqrt{2}$.
20. $V_a : V_b = b : a$.
21. $S = 6\pi a^2\sqrt{3}$; $V = \frac{9\pi a^3}{2}$.
22. 4 : 3.
23. $16 : 7\sqrt{3}$.
24. 60°.
25. $S = 4\pi\sqrt{3}$ sq. ft.; $V = 6\pi$ cub. ft.
26. $V = \pi r^3$.
27. 3 : 2.
28. 9 : 4 : 32.
29. They are equal.
30. 3 : 2 : 1.

§ 11. Inscribed and Circumscribed Solids.

1. 80 cub. ft.
2. 616 sq. ft.
3. 3 : 2 : 1.
4. $3a : 2b : a$.
5. 14 : 11.
6. 11 : 7.

7. $5\frac{1}{2}$ lbs.

8. $r=\frac{a}{12}\sqrt{6}$; $r'=\frac{a}{4}\sqrt{6}$; $S=a^2\sqrt{3}$; $V=\frac{a^3}{12}\sqrt{2}$.

9. $r=\frac{a}{6}\sqrt{6}$; $r'=\frac{a}{2}\sqrt{2}$; $S=2a^2\sqrt{3}$; $V=\frac{a^3}{3}\sqrt{2}$.

10. $\sqrt{3}:9$.

11. $1:8$.

12. $1:2\sqrt{2}$.

13. $\frac{1}{3}h(r^2+r'^2+rr')$.

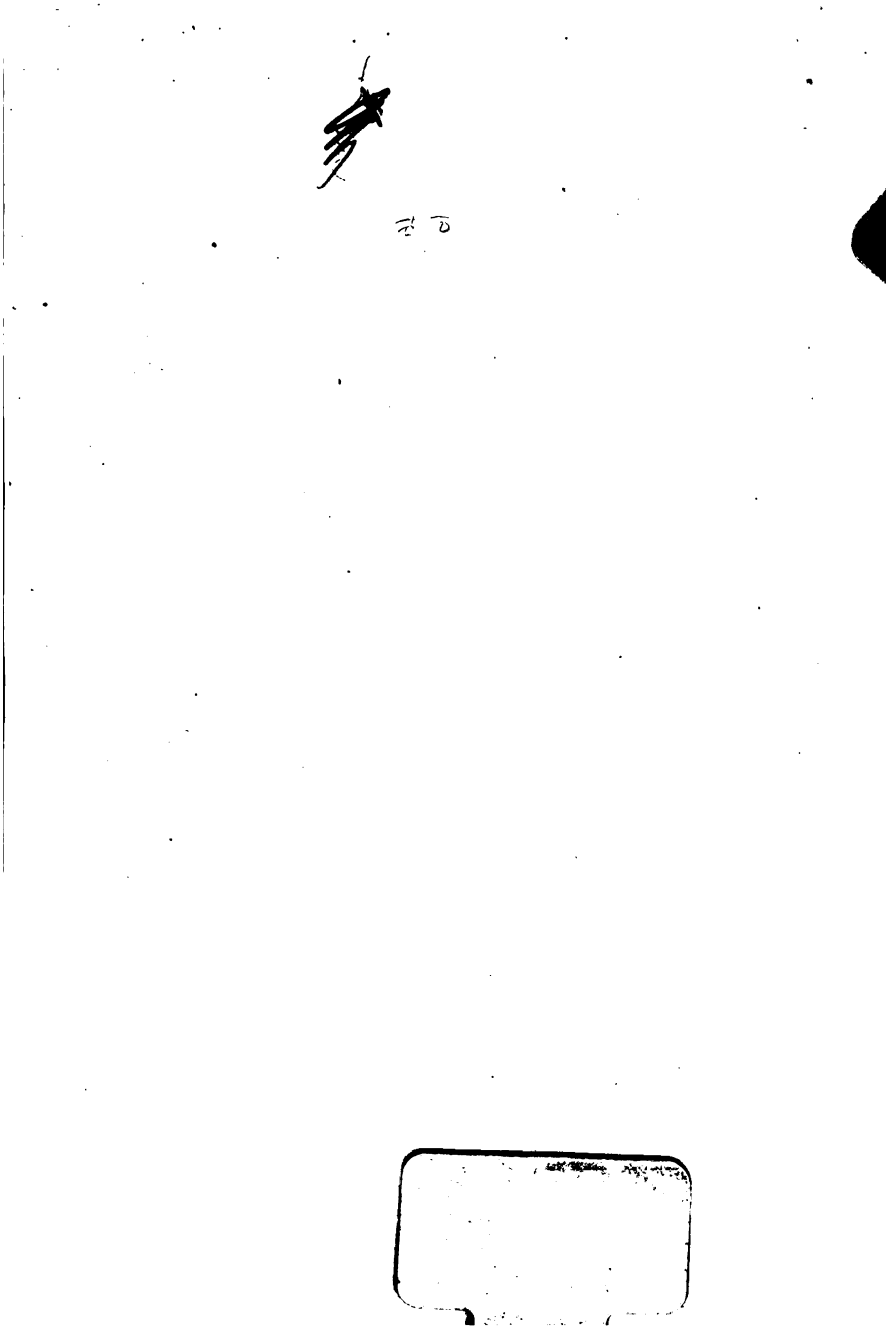
14. $\frac{3V}{\pi}\sqrt{3}$.

15. $1:8$.

16. $\frac{2\pi a^4 b}{3(a^2-b^2)}$.

17. $\frac{V\sqrt{3}}{2\pi}$.

18. $\frac{1}{4}\pi V$.

19. $2S$.

20. $r\sqrt{3}$.